The Hubble expansion and the cosmic redshift

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Here we derive a few preliminary expression for the redshift.

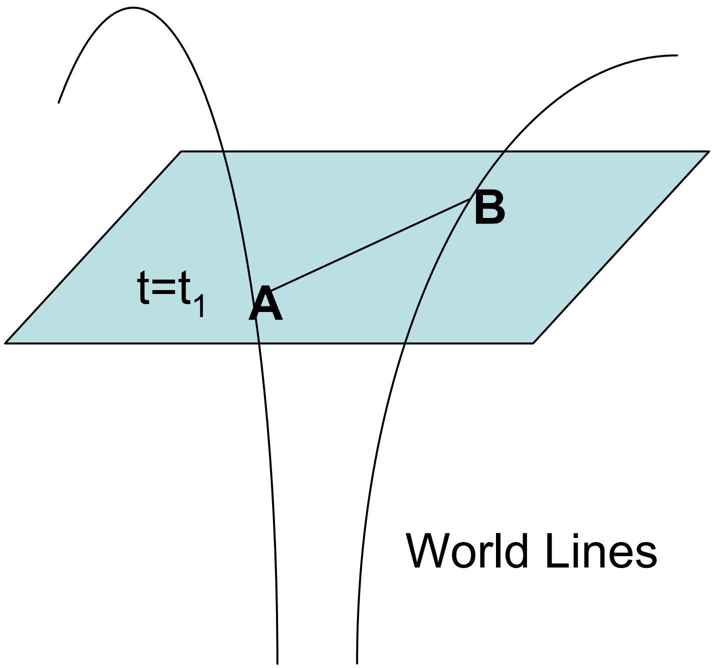
Try to see the difference between Doppler redshift and Cosmological redshift. The difference between the signal received from a distant object in an expanding Universe and an object moving away from us respect our frame of reference in the Laboratory.

The events

- Time like
  - $ds^2 > 0$. For instance I have two events at the same location. In this case $ds^2 = c^2 dt^2$.
- Space like
  - $ds^2 < 0$. Two events occurr at the same time, $dt=0$, in different locations and we have $ds^2 = -dl^2$.
- Light like
  - $ds = 0$. Two points can be connected by a light signal.
  - $ds^2 = c^2 dt^2 - dl^2$.

Distance between two points

- I measure at the same time so that dt=0.
- For convenience I use r rather than σ in the Robertson Walker metric we just derived. I indicate by d the distance.



$$d = R(t)\int_0^{r_1}\frac{dr}{\sqrt{1-kr^2}}\ ;\ k\begin{cases}+1\\0\\-1\end{cases}$$

$$\dot{d}(t) = \dot{R}(t)\int_0^{r_1}\frac{dr}{\sqrt{1-kr^2}} = \dot{R}(t)\frac{R}{R}\int_0^{r_1}\frac{dr}{\sqrt{1-kr^2}}$$

$$\dot{d}(t) = \frac{\dot{R}(t)}{R(t)}d \Rightarrow V = H(t)d$$

I choose a system of coordinates For which $\Delta\Theta=0$, $\Delta\phi = 0$.

Always A & B

At the time t A & B are separated by l(t) proper distance and that is a function of the time t.

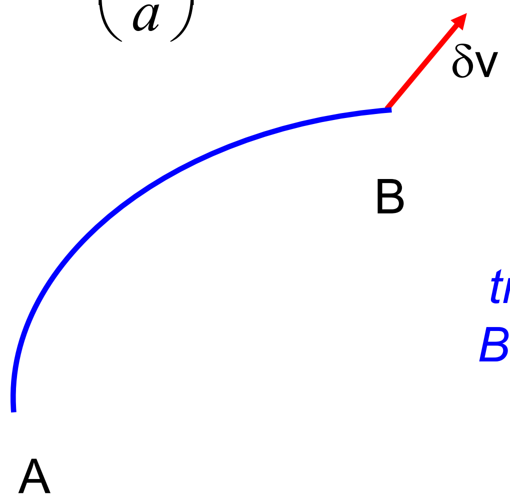
$$l(t) = l_0 R(t) \propto R(t) \equiv \delta l = a(t)\,\delta x$$

$$\delta v = \frac{\partial}{\partial t}\delta l = \delta x \frac{\partial}{\partial t} a(t) = \dot{a}\,\delta x = \frac{\dot{a}}{a} a\,\delta x$$

$$\delta v = \left(\frac{\dot{a}}{a}\right)\delta l$$

l is the p[roper separation
L_0 the separtion (comoving)

This I could interpret as the Hubble law

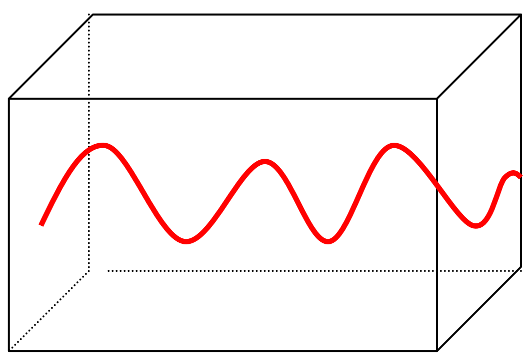


Now assume that both observers see the transit of a photon. The photon takes tp go from A to B a transit time $\delta t = \delta l/c$ and the observer B is moving relative to the first observer with a speed δv.

Since δl is small I use the Doppler. Furthermore I define as δa the Variation of a in the time δt. I can write:

$$\frac{\delta\lambda}{\lambda} = \frac{\delta v}{c} = \frac{1}{c}\left(\frac{\dot{a}}{a}\right)\delta l = \left(\frac{\dot{a}}{a}\right)\delta t = \frac{\delta a}{a}\ ;\ int\,egrating$$

$$ln\ \lambda = ln\ a\ ;\ or\ \frac{\lambda}{a} = const\ ;\ \frac{\lambda_1}{\lambda_2} = \frac{a_1}{a_2}$$

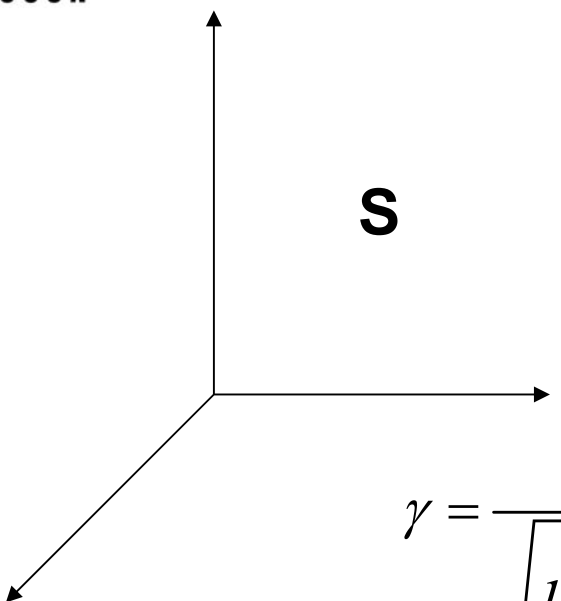


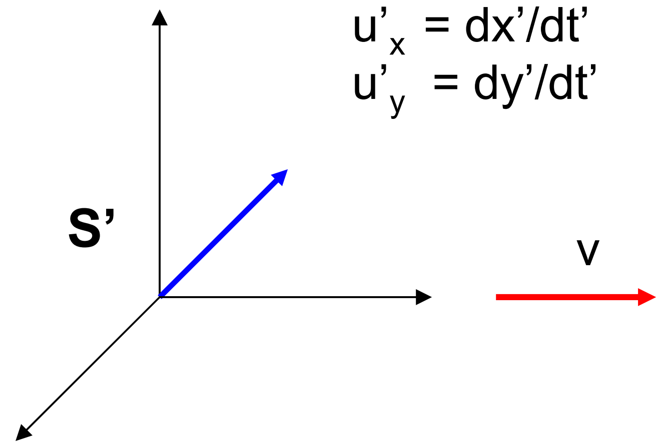
If the extremety of the wave are attached to the sides of the box by making the box larger or smaller I change the wavelength.

And what about if a particle transit the two observers? A sees a particle passing by with a certain velocity v and however B is moving relative to A with a velocity that we call now δu = (á/a) δl.

We also remind that the particle moves rather fast. Lorentz Transformations

Lorentz Transformation - Coordinates





$u'_x = dx'/dt'$
$u'_y = dy'/dt'$

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$x' = \gamma\,(x - vt) \qquad x = \gamma\,(x' - vt')$$

$$y' = y \qquad y = y'$$

$$z' = z \qquad z = z'$$

$$t' = \gamma\left(t - \frac{v\,x}{c^2}\right) \qquad t = \gamma\left(t' - \frac{v\,x'}{c^2}\right)$$

S measure S' moving with velocity v along the x axis

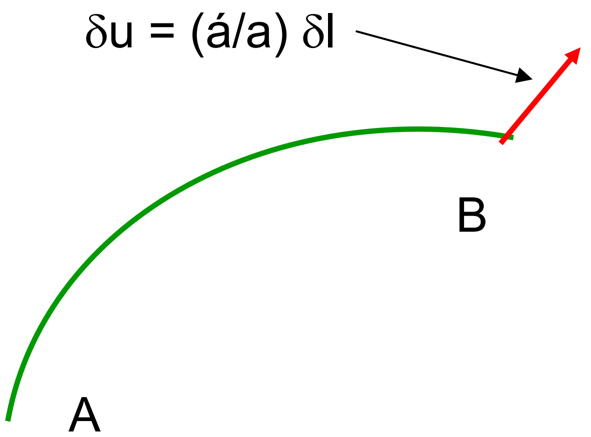
Lorentz Transformation - Velocities

$$dx = \gamma\left(\frac{dx'}{dt'} + v\right)dt' \;; dy = \frac{dy'}{dt'}dt' \;; dt = \gamma\left(1 + v\frac{dx'/dt'}{c^2}\right)dt'$$

$$\frac{dx}{dt} = u_x = \frac{u'_x + v}{dt}dt' = \frac{u'_x + v}{1 + v\frac{dx'/dt'}{c^2}} = \frac{u'_x + v}{1 + v\frac{u'_x}{c^2}}$$

$$\frac{dy}{dt} = u_y = \frac{u'_y/\gamma}{1 + v\frac{u'_x}{c^2}}$$

$$u'_x = \frac{u_x - v}{1 - v\frac{u_x}{c^2}} \quad ; \quad u'_y = \frac{u_y/\gamma}{1 - v\frac{u_x}{c^2}}$$



$$\delta u = \left(\frac{\dot a}{a}\right)\delta l = \left(\frac{\dot a}{a}\right) v\,\delta t = v\frac{\delta a}{a}$$

for the second observer

$$v' = \frac{v - \delta u}{1 - v\frac{\delta u}{c^2}} = \left|Taylor\ series\ in\ \delta u\right| \Rightarrow f(0) = v$$

$$f'(0) = \frac{\left(1 - v\frac{\delta u}{c^2}\right)(-1) - (v - \delta u)\left(-\frac{v}{c^2}\right)}{\left(1 - v\frac{\delta u}{c^2}\right)^2} = \frac{\frac{v^2}{c^2} - 1}{\left(1 - v\frac{\delta u}{c^2}\right)^2} = \frac{v^2}{c^2} - 1$$

$$v' = v + \left(\frac{v^2}{c^2} - 1\right)\delta u + O\left(\delta u^2\right) = v - \left(1 - \frac{v^2}{c^2}\right) v \frac{\delta a}{a}$$

$$v' - v = \delta v = -v\left(1 - \frac{v^2}{c^2}\right)\frac{\delta a}{a}$$

Integrating

$$\delta v = -v\left(1-\frac{v^2}{c^2}\right)\frac{\delta a}{a} \Rightarrow \frac{\delta v}{v\left(1-\frac{v^2}{c^2}\right)} = \frac{\delta a}{a}$$

$$\frac{v}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma\, v = \frac{const}{a} \Rightarrow \gamma_1 v_1 a_1 = \gamma_2 v_2 a_2$$

$$Photon \quad v \to c \quad \gamma\, v \to h\,\nu$$

$$h\,\nu_1 a_1 = h\,\nu_2 a_2 \Rightarrow \frac{\nu_1}{\nu_2} = \frac{a_2}{a_1} = \frac{\lambda_1}{\lambda_2} \text{ as done earlier}$$

The concept of Pressure

- If for the galaxies I write $\varepsilon = \rho c^2$ where ρ is the mass density I have:

$$P \propto \rho T \propto \rho v^2 c^2/c^2 \propto \varepsilon v^2/c^2$$

- I could estimate at the time $t=t_0$ (now) the rms velocity of the galaxies. This is in the range 50 to 1000 km/s going from the field to clusters of galaxies. For larger Structures as Super_clusters, not virialized however, we have a dispersion in velocity (~infall) of about 500 km/s. In this case:

$$P \sim 10^{-5}\, 10^{-6} \text{ very small } p \sim 0$$

- On the other hand we have seen $\gamma v \propto 1/a(t)$ which may become very large for a(t) small. In the past the motions of galaxies should have been very turbolent and v rather high. In this case the pressure term could be $\neq$ from zero. However:
- The galaxies form and evolve when a(t) is smaller. That is before considering the above relation between p & ε I must understand the physics at that epoch.
- If I simply extrapolate the relation $\gamma v \propto 1/a(t)$ at a point where $v \sim c$ my result breaks down.
- In the relativistic domain I must use $p = 1/3\, \varepsilon$.

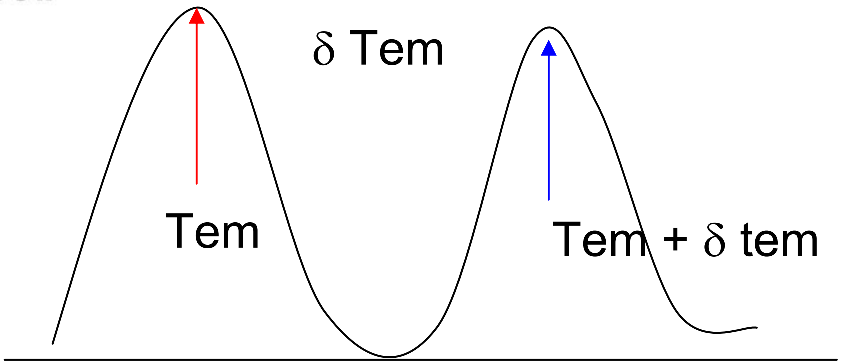
Pressure

see Vol I Relativistic Astrophysics by Zeldovich and Novikov

- $\rho < 50$ g cm^{-3}
  - There are pronounced individual and chemical properties of the elements which change from one element to the next in accordance with Mendeleev's periodic law.
- $50 < \rho < 500$
- $500 < \rho < 10^{11}$
- $10^{11} < \rho < 10^{14}$
- $10^{14} < \rho < 10^{16}$
- $10^{16} < \rho < 10^{93}$
- $\rho > 10^{93}$

- A student will work on this or I should be reminded to prepare for it.

Cosmological redshift

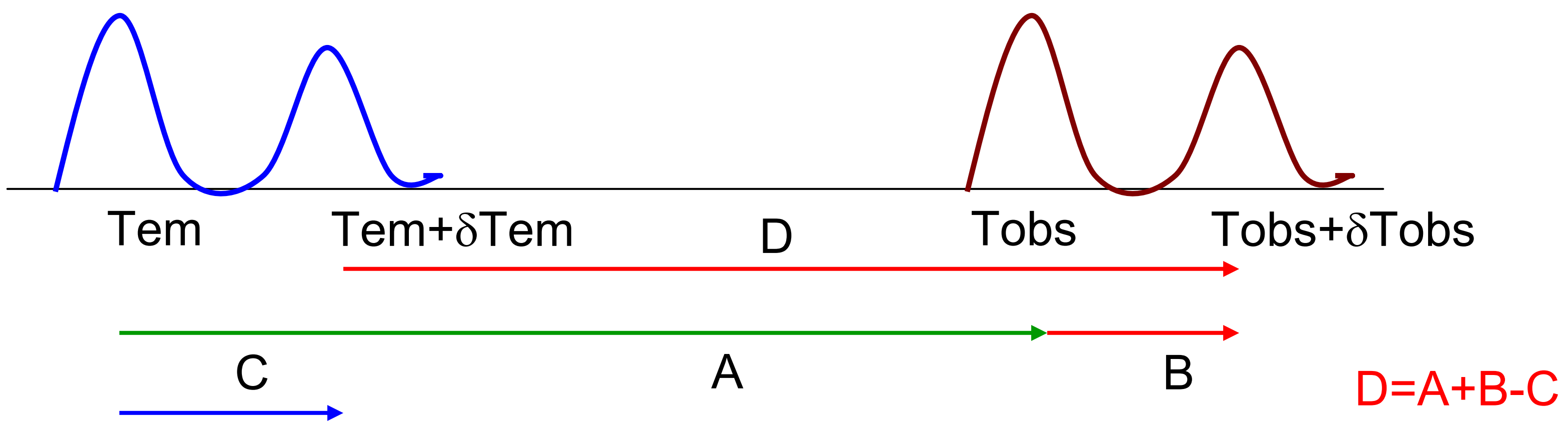


Photon } $ds^2 = 0$, $d\theta = 0$, $d\varphi = 0$

$$0 = c^2 dt^2 - a^2(t)\frac{dr^2}{1-kr^2}$$

$$c\int_{Tem}^{Tobs}\frac{dt}{a(t)} = \int_0^{r_1}\frac{dr}{\sqrt{1-kr^2}} \Leftarrow \frac{c\,dt}{a(t)} = \frac{dr}{\sqrt{1-kr^2}}$$

Transform the Integral



$$\int_{T_{em}+\delta T_{em}}^{T_{obs}+\delta T_{obs}} \frac{dt}{a(t)} = \int_{T_{em}}^{T_{bs}} + \int_{T_{obs}}^{T_{obs}+\delta T_{obs}} - \int_{T_{em}}^{T_{em}+\delta T_{em}} \quad ; c\int_{T_{em}}^{T_{obs}} \frac{dt}{a(t)} = \int_0^{r_1} \frac{dr}{\sqrt{1-kr^2}}$$

$$and\ c\int_{T_{em}+\delta T_{em}}^{T_{obs}+\delta T_{obs}} \frac{dt}{a(t)} = \int_0^{r_1} \frac{dr}{\sqrt{1-kr^2}}\ I\ have: \int_{T_{em}}^{T_{bs}} \frac{dt}{a(t)} = \int_{T_{em}+\delta T_{em}}^{T_{obs}+\delta T_{obs}} \frac{dt}{a(t)}$$

Therefore the equation reduces to:

$$\int_{T_{obs}}^{T_{obs}+\delta T_{obs}} \frac{dt}{a(t)} = \int_{T_{em}}^{T_{em}+\delta T_{em}} \frac{dt}{a(t)}$$

And Finally

I assume that a(t) ≫ δ(t) so that a(t) does not change (=small and I can disregard it) during the interval δt. That is I can write:

$$\frac{1}{a(T_{obs})} * \int_{T_{obs}}^{T_{obs}+\delta T_{obs}} dT = \frac{1}{a(T_{em})} * \int_{T_{em}}^{T_{em}+\delta T_{em}} dT$$

$$\frac{\delta T_{obs}}{a(T_{obs})} = \frac{\delta T_{em}}{a(T_{em})}; \delta T = \frac{1}{\nu} = \frac{\lambda}{c}$$

$$\frac{\lambda_{em}}{a(T_{em})} = \frac{\lambda_{obs}}{a(T_{obs})} \ and\ note:$$

$$1+z = \frac{\lambda_{obs}}{\lambda_{em}} = \frac{a(T_{obs})}{a(T_{em})}$$