Galaxy Formation and non Linear collapse
Part I

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Cosmology Lectures

This part follows to a large extent Padmanabhan

General Considerations

We have seen that the Universe is composed by three main components: Dark Matter, Baryonic Matter and Photons.

As we have seen the Nucleo-synthesis puts very stringent limits on the amount of Baryons we have in the Universe. And we have $0.011 <= \Omega_B h^2 <= 0.037$.

Dynamical Measurements on the Mass of clusters of Galaxies and considerations over the Large Scale distribution of matter call do a density of about $\Omega_{DM} \sim 0.25$ (with h=0.7 the baryonic contribution to Ω is about 0.05). The observational data will be discussed later on.

The observations of the SNe (see dedicated lecture) and the observations of the MWB show that $\Omega_{Tot} = 1$

These three components must be treated each one in a different way since their characteristics are completely different.

Dark Matter is the simplest one. It is collisionless (see however other points of view) and affected only by gravity.

The complication with photons is that they can traverse a distance of the order of the Horizon in a cosmic time. That is they tend to wipe out perturbations because of their free propagation.

Baryonic Matter present difficulties due to the fact it is coupled to photons and by all those mechanism that involve dissipation and cooling.

Finally the equation to be solved also for DM are rather complicated and that is why numerical simulations are rather useful. On the other hand when the perturbation are small it is possible to use the linear approximation and make it simpler. Or for the non linear collapse the assumption of spherical symmetry helps a lot.

In this phenomenological description it is clear for instance that baryonic perturbations can not grow while matter and photons are coupled because the photon field tend to stop the growing process.

On the other hand DM perturbations that do not feel the action by the photons can grow much earlier. It is clear therefore that perturbations in the DM will induce corresponding perturbations in the baryons since, as we have seen, the mass of the DM is about 10 times the mass of the BM.

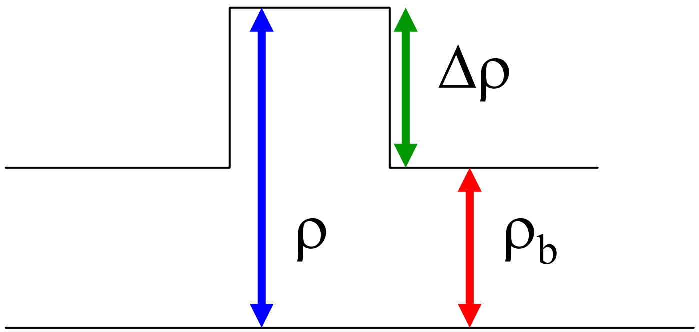
Note also that we never mentioned about the seed of the perturbations. These must be present some how ab initio.

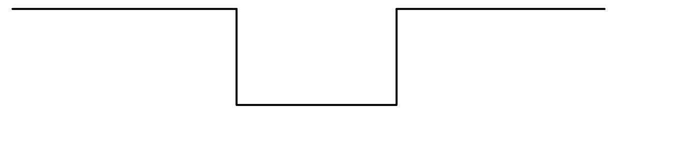
We now see in a simplified way some of the processes occurring and leading to the formation of galaxies.

Density perturbations

- We have seen that under particular condition the perturbation densities grow and after collapse my generate, assuming they reach some equilibrium, an astronomical object the way we know them.
- Density perturbations may be positive, excess of density, or negative, deficiency of density compared to the background mean density.
- Now we must investigate two directions:
  - The spectrum of perturbations, how it is filtered though the cosmic time and how it evolves and match the observations.
  - How a single perturbation grow or dissipate and which are the characteristic parameters as a function of time.
- Here we will be dealing with the second problem and develop next the formation and evolution of the Large Scale structure after taking in consideration the observations and the methods of statistical analysis.

Visualization





$$\frac{\Delta\rho}{\rho_b} = \frac{\rho - \rho_b}{\rho_b} = \frac{\rho}{\rho_b} - 1 = \delta \succ 0 \qquad \frac{\Delta\rho}{\rho_b} = \frac{\rho - \rho_b}{\rho_b} = \frac{\rho}{\rho_b} - 1 = \delta \prec 0$$

$$\Phi_{Tot} = \Phi_{Background} + \delta\Phi_{Perturbation}$$

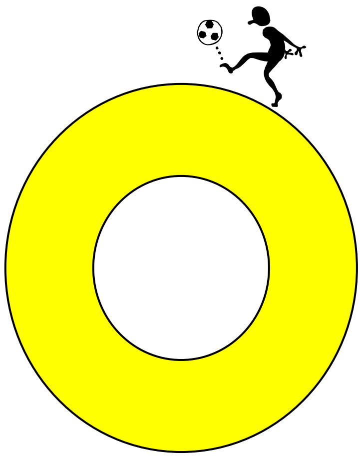
$$\rho(r,t_i) = \rho_b(t_i)\big(1 + \delta(r,t_i)\big) = \rho_b(t_i) + \rho_b(t_i)\delta(r,t_i) = \rho_b(t_i) + \Delta\rho$$

Definition of the problem

- We use proper radial coordinates $r = a(t)\ x$ in the Newtonian limit developed in class and where x is the co_moving Friedmann coordinate. Here we will have:
- φ_b = Equivalent potential of the Friedmann metric
- $\delta\ \varphi(r,t)$ = The potential generated due to the excess density: $\Delta\rho$
- It is then possible to demonstrate, see Padmanabahn Chapter 4, that the first integral of motion is:

$$\tfrac{1}{2}(dr/dt)^2 - GM/r = E$$

How would it move a particle on a shell?

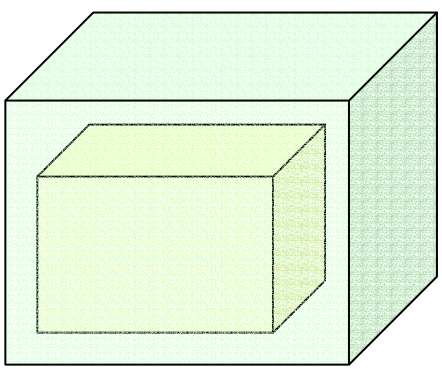


$$Force = -\nabla \Phi$$

$$\frac{d^2 r}{dt^2} = -\nabla \Phi_{Total} = \frac{\partial}{\partial r}\frac{GM_{Tot}}{r} = -\frac{4\pi G\rho_b(t)}{3}\bar{r} - \nabla(\delta\Phi) =$$

$$= \frac{GM_b}{r^3}\bar{r} - \frac{G\delta M(r,t)}{r^3}\bar{r}$$

The Universe Expands



$$r(t) = a(t)\,x$$

$$M_b = \frac{4}{3}\pi\rho_b(t)r^3 = \frac{4}{3}\pi\rho_b(t)\left(a(t)\,x\right)^3$$

$$\delta M_b(r,t) = 4\pi\int_0^r \Delta\rho(q,t)\,q^2 dq = 4\pi\rho_b(t)\int_0^r \Delta\rho(q,t)\,q^2 dq$$

And we can look at a series of shells and also assume that the shells contracting do not cross each other

Mass within any shell at the initial value i

$$\mathfrak{M}_{Tot} = \mathfrak{M}_i + \rho_b\left({}^4\!/_3\,\pi r_i^3\right) = \int_0^{r_i} 4\pi r^2 \Delta_i dr + \rho_b\left({}^4\!/_3\,\pi r_i^3\right)$$

$$\bar{\delta}_{ti} \equiv \bar{\delta}_i = \frac{1}{\rho_b}\frac{\int_0^{r_i} 4\pi r^2 \Delta_i dr}{{}^4\!/_3\,\pi r^3} = \frac{\mathfrak{M}_i(\,pert.)}{Volume}\frac{1}{\rho_b} = \frac{\rho}{\rho_b} - 1$$

$$\mathfrak{M} = \mathfrak{M}_i + \rho_b\left({}^4\!/_3\,\pi r_i^3\right) = \rho_b\left({}^4\!/_3\,\pi r_i^3\right)(1+\bar{\delta}_i)$$

Remember

- At a well defined time t_x I have a well set system of coordinates and each object has a space coordinate at that time. I indicate by x the separation between two points.
- If at some point I make the Universe run again, either expand or contract, all space quantities will change accordingly to the relation we found for the proper distance etc. That is r (the proper separation) will change as a(t) x.
- Or $a(t_x)\ x = a(t)\ x$ and in particular:
- $r_o = a(t_o)\ x = a(t)\ x = r \Rightarrow r/r_o = a(t)/a(t_o)$ and for $r_o = x$
    - $x = r\ a(t_o)/a(t)$.

Situation similar to the solution of Friedman equations

½(dr/dt)2 – GM/r =E

$$E > 0 \Rightarrow \dot{r}_i^2 \; always > 0 \; the \; shell \; exp ands \; foreverr$$

$$E < 0 \Rightarrow \dot{r}_i^2 \; could \; be \; zero \; or \; negative \Rightarrow collapse$$

$$At \; t_i \; I \; assume \; a \; small \; overdensity \; and \; the \; shell \; exp ands \; with$$

$$the \; background$$

$$\dot{r}_i = \dot{a}\,x = \frac{\dot{a}}{a} a\,x = \frac{\dot{a}}{a} r_i = H\left(t_i\right) r_i = H_i r_i$$

$$K_i = \frac{\dot{r}_i^2}{2} = \frac{H_i^2 r_i^2}{2}$$

$$|U| = \left(\frac{GM}{r}\right)_{t_i} = \rho_b\left(t_i\right)\left(\tfrac{4}{3}\pi r_i^2\right)\left(1 + \bar{\delta}_i\right) \;\&\; \rho_b\left(t_i\right) = \frac{3H_i^2}{8\pi G}\Omega_i$$

$$\left|U\right| = \rho_b\left(t_i\right)\left(\tfrac{4}{3}\pi r_i^2\right)\left(1+\bar{\delta}_i\right) = \frac{3H_i^2}{8\pi G}\Omega_i\left(\tfrac{4}{3}\pi r_i^2\right)\left(1+\bar{\delta}_i\right) =$$

$$= \frac{H_i^2 r_i^2}{2}\Omega_i\left(1+\bar{\delta}_i\right) = K_i\Omega_i\left(1+\bar{\delta}_i\right) \quad \&$$

$$E = K_i - K_i\Omega_i\left(1+\bar{\delta}_i\right) = K_i\Omega_i\left[\Omega_i^{-1} - \left(1+\bar{\delta}_i\right)\right]$$

That is collapse for

$$E < 0 \Rightarrow \left(1+\bar{\delta}_i\right) > \Omega_i^{-1} \Rightarrow \bar{\delta}_i > \Omega_i^{-1} - 1$$

And remember that Ω_I was defined in relation to the Background surrounding the pertubation at the time t_i

For the case of interest E <0
we have collapse when:

$$\bar{\delta}_i \succ [\Omega^{-1}{}_i - 1]$$

Case A

$$\Omega_i^{-1} > 1$$

$$\Omega_i < 1$$

$$\delta_i > \delta_{crit} = \Omega_i^{-1}$$

Case B

$$\Omega_i^{-1} \leq 1$$

$$\Omega_i \geq 1$$

$$\delta_i > 0$$

Always verified

$$\frac{r_m}{r_i} = \frac{1 + \bar{\delta}_i}{\bar{\delta}_i - (\Omega_i^{-1} - 1)}$$

Derivation of r_m/r_i

- The overdensity expands together with the background and however at a slower rate since each shell feels the overdensity inside its radius and its expansion is retarded. Perturbation in the Hubble flow caused by the perturbation.
- The background decreaseds faster and the overdensity grow to a maximum radius r_m at which point the collapse begins for an overdensity larger than the critical overdensity as stated in the previous slide.

$$At\ this\ point\ \dot{r}_i = 0,\ therefore\ K_m = 0.0$$

$$E_m = -\frac{GM}{r_m} = -\frac{GM}{r_m}\frac{r_i}{r_i} = --\frac{GM}{r_i}\frac{r_i}{r_m} = \left\{ |U| = K_i \Omega_i \left(1+\bar{\delta}_i\right) \right\} =$$

See slides N. 8, N.9

$$= -\frac{r_i}{r_m} K_i \Omega_i \left(1+\bar{\delta}_i\right) = E_i\left(Energy\ conservation\right)$$

$$-\frac{r_i}{r_m} K_i \Omega_i \left(1+\bar{\delta}_i\right) = E_i = K_i \Omega_i \left[\Omega_i^{-1} - \left(1+\bar{\delta}_i\right)\right]$$

$$\frac{r_m}{r_i} = -\frac{1+\bar{\delta}_i}{\Omega_i^{-1} - \left(1+\bar{\delta}_i\right)} = \frac{1+\bar{\delta}_i}{\bar{\delta}_i - (\Omega_i^{-1} - 1)}$$

Generalities

At the very early epochs when the modes of a given scale-lengths are outside the Horizon the perturbations are not affected by the microphysics and all components, baryonic and non, grow more or less in the same manner.

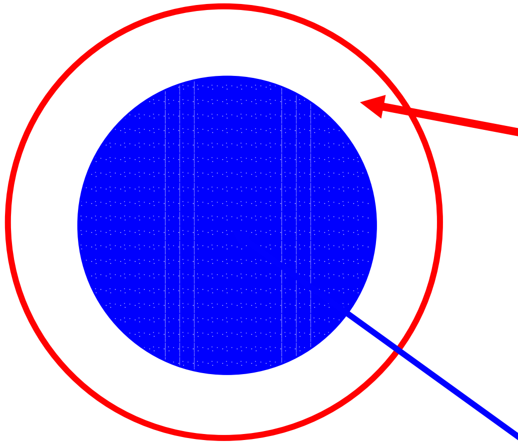
While formally we can not use the Newtonian perturbation theory when the scale length is larger than the Horizon we can estimate functional dependence of the linear growth of the perturbations using this very simple model.

Consider a large ($\lambda >> r_H$) perturbation embedded in a Friedman Universe with k=0 and background density ρ_b. For simplicity we also assume spherical symmetry. Outermost shell k=0, innermost $k \neq 0$.

In this model the innermost region , the perturbation is not affected by what happens in the outermost region and it evolves independently.

Under these assumption dark Matter and baryonic matter perturbations grow equally and however as soon as the perturbation enter the Horizon the microphysics will prevent the growth of baryonic perturbations. See previous discussion on the Jean Mass as a function of cosmic time.

We now put these very simple concepts in a more quantitative way. We will also see that linear growth will degenerate in non linear growth as soon as the density contrast is above a given limit.



$$H^2 = \frac{8\pi G}{3}\rho_b$$

$$H^2 = \frac{8\pi G}{3}\left(\rho_b + \Delta\rho\right) - \frac{k\,c^2}{a^2}$$

The change of density is accommodated by changing curvature and to maintain this condition at all time

$$\frac{8\pi G}{3}\Delta\rho = \frac{k\,c^2}{a^2} \quad \Rightarrow \quad \delta\rho = \frac{\Delta\rho}{\rho_b} = \frac{3}{8\pi G}\frac{k\,c^2}{a^2}\frac{1}{\rho_b}$$

$$\delta\rho = \frac{\Delta\rho}{\rho_b} \propto \begin{cases} \left(for\ t < t_{eq}\ \rho_b \propto a^4\right) \Rightarrow a^2 \\ \left(for\ t > t_{eq}\ \rho_b \propto a^3\right) \Rightarrow a \end{cases}$$

And at a later time the perturbation enters the Horizon

The Perturbation evolves

And shells do not cross and I conserve the mass

From the equations of motion – compare with Friedman equations

$$r = A(1-\cos\theta)$$

$$t = B(\theta - \sin\theta) \;\; ; \;\; A^3 = G\mathfrak{M}B^2$$

$$\bar{\rho}(r,t) = \frac{3\mathfrak{M}}{4\pi r^3} = \frac{3\mathfrak{M}}{4\pi A^3(1-\cos\theta)^3} \; ; \quad \textit{and for } \Omega_i \equiv \Omega = 1$$

$$a \propto t^{\frac{2}{3}} \quad \rho_b = \frac{1}{6\pi G t^2} = \frac{1}{6\pi G B^2(\theta - Sin\theta)^2} \textit{ so that}$$

$$\frac{\bar{\rho}(r,t)}{\rho_b(t)} = 1 + \bar{\delta}(r,t) = \frac{3\,\mathfrak{M}\,6\pi G\,B^2(\theta - Sin\theta)^2}{4\pi A^3(1-\cos\theta)^3} = \frac{3\,6\pi\,(\theta - Sin\theta)^2}{4\pi(1-\cos\theta)^3} = \frac{9}{2}\frac{(\theta - Sin\theta)^2}{(1-\cos\theta)^3}$$

$$\bar{\delta}(r,t) = \frac{9}{2}\frac{(\theta - Sin\theta)^2}{(1-\cos\theta)^3} - 1 \Rightarrow \bar{\rho}(t) = \rho_b(t)\frac{9}{2}\frac{(\theta - \sin\theta)^2}{(1-\cos\theta)^3}$$

A & B

$$For\ r = r_m \quad \theta = \pi \quad r_m = 2\,A\ and\ because\ \frac{r_m}{r_i} = \frac{1+\overline{\delta}_i}{\overline{\delta}_i - (\Omega_i^{-1} - 1)}$$

$$A = \frac{1}{2} r_i \frac{1+\overline{\delta}_i}{\overline{\delta}_i - (\Omega_i^{-1} - 1)}\ and$$

$$B^2 = \frac{A^3}{G\,\mathfrak{M}} = \left(\frac{1}{2} r_i \frac{1+\overline{\delta}_i}{\overline{\delta}_i - (\Omega_i^{-1} - 1)}\right)^3 \frac{1}{G\rho_b \left(\frac{4}{3}\pi r_i^3\right)\left(1+\overline{\delta}_i\right)} =$$

$$= \left(\frac{1}{2} r_i \frac{1+\overline{\delta}_i}{\overline{\delta}_i - \left(\Omega_i^{-1} - 1\right)}\right)^3 \frac{1}{G\Omega_i \frac{3H_i^2}{8\pi G}\left(\frac{4}{3}\pi r_i^3\right)\left(1+\overline{\delta}_i\right)} = \frac{\left(1+\overline{\delta}\right)^2}{4\,\Omega_i H_i^2 \left(\overline{\delta}_i - \left(\Omega_i^{-1} - 1\right)\right)^3}$$

$$B = \frac{\left(1+\overline{\delta}_i\right)}{2\,\Omega_i^{\frac{1}{2}} H_i \left(\overline{\delta}_i - \left(\Omega_i^{-1} - 1\right)\right)^{\frac{3}{2}}}$$

Starting with small perturbations

$$Sin\,\theta \simeq \theta - \frac{1}{6}\theta^3 + \ldots.. \qquad Cos\,\theta = 1 - \frac{\theta^3}{2} + \ldots..$$

$$\bar{\delta}(r,t) = \frac{9}{2}\frac{(\theta - Sin\theta)}{(1-\cos\theta)^3} - 1 \simeq \frac{3\theta^2}{20} + \frac{37\,\theta^4}{2800} + O[\theta]^6 \simeq \frac{3\theta^2}{20}$$

$$t = B(\theta - \sin\theta) \simeq B\frac{\theta^3}{6} \qquad \theta \simeq \left(\frac{6\,t}{B}\right)^{\frac{1}{3}} \qquad \bar{\delta}(r,t) \simeq \frac{3\left(\frac{6\,t}{B}\right)^{\frac{2}{3}}}{20}$$

$$\bar{\delta}(r,t) \simeq \frac{3\left(\frac{6t}{B}\right)^{\frac{2}{3}}}{20} = \frac{3}{20}(6t)^{\frac{2}{3}}\left\{\frac{\left(1+\bar{\delta}_i\right)}{2\Omega_i^{\frac{1}{2}}H_i\left(\bar{\delta}_i-\left(\Omega_i^{-1}-1\right)\right)^{\frac{3}{2}}}\right\}^{-\frac{2}{3}} \simeq \left(for\ \Omega_1 = 1; t_i = \frac{2}{3}H^{-1}{}_i\right) \simeq$$

$$\frac{3}{20}\left\{\frac{\left(1+\bar{\delta}_i\right)}{6t\,2\frac{2}{3}\frac{1}{t_i}\left(\bar{\delta}_i\right)^{\frac{3}{2}}}\right\}^{-\frac{2}{3}} = \frac{3}{20}\left\{\frac{4}{3}\frac{6t\left(\bar{\delta}_i\right)^{\frac{3}{2}}}{t_i}\right\}^{\frac{2}{3}} = \frac{3}{20}8^{\frac{2}{3}}\bar{\delta}_i\left(\frac{t}{t_i}\right)^{\frac{2}{3}}$$

$$\bar{\delta}(r,t) \simeq \frac{3}{5}\bar{\delta}_i\left(\frac{t}{t_i}\right)^{\frac{2}{3}} \propto a(t) \propto \frac{1}{(1+z)}$$

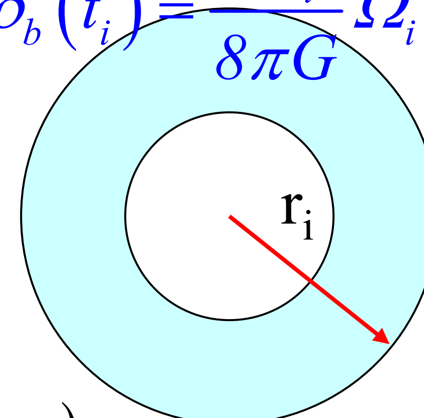
3/5 of the perturbation is in the growing mode and this is the growth in the linear regime which could be compared to the non linear growth. We did that as an approximation for small perturbation but we could develop the equation in linear regime for small perturbations.

$$\bar{\delta}(r,t) \simeq \frac{3}{5}\bar{\delta}_i \left(\frac{t}{t_i}\right)^{\frac{2}{3}} \propto a(t) \propto \frac{1}{(1+z)} \quad \{for\ any\ t\} \quad \bar{\delta}(r,t) \simeq \frac{3}{5}\bar{\delta}_i \frac{(1+z_i)}{(1+z)}\ or$$

in a different way for $\Omega_i = 1$ *and* $\bar{\delta}_i$ *small*

$$\rho_b(t_i) = \frac{3H_i^2}{8\pi G}\Omega_i$$

$$\left.\begin{aligned} A &= \frac{1}{2} r_i \frac{1+\bar{\delta}_i}{\bar{\delta}_i} \\ B &= \frac{(1+\bar{\delta}_i)}{2H_i(\bar{\delta}_i)^{\frac{3}{2}}} \end{aligned}\right\} \Rightarrow \begin{aligned} A &= \frac{1}{2} r_i \frac{1}{\bar{\delta}_i} = \frac{1}{2}\frac{x}{(1+z_i)}\frac{3}{5}\frac{(1+z_i)}{\delta_0} = \frac{3}{10}\frac{x}{\delta_0} \\ B &= \frac{3}{4}\frac{t_i}{(\bar{\delta}_i)^{\frac{3}{2}}} = \frac{3}{4}\left(\frac{3}{5}\frac{(1+z_i)}{\delta_0}\right)^{\frac{3}{2}} \frac{t_0}{(1+z_i)^{\frac{3}{2}}} = \left(\frac{3}{5}\right)^{\frac{3}{2}} \frac{3}{4}\frac{t_0}{\delta_0^{\frac{3}{2}}} \end{aligned}$$



$$r_i = \frac{a(t_i)}{a_0}x = \frac{x}{(1+z_i)}\ and\ \ I\ define\ from\ above\ \delta_0 = \frac{3}{5}\bar{\delta}_i(1+z_i)$$

If at the redshift z_i I had a density contrast δ_i the present value would be δ_0.

Using the approximations for A & B

$$r = A\,(1-\cos\theta) = \frac{3}{10}\frac{x}{\delta_0}(1-\cos\theta)$$

$$t = B\,(\theta-\sin\theta) = \left(\frac{3}{5}\right)^{\frac{3}{2}}\frac{3}{4}\frac{t_0}{\delta_0^{\frac{3}{2}}}(\theta-\sin\theta)$$

$$\bar{\rho}(r,t) = \rho_b(t)\frac{9}{2}\frac{(\theta-\sin\theta)^2}{(1-\cos\theta)^3}$$

$$(1+z) = (t/t_0)^{-2/3} = \frac{5}{3}\left(\frac{4}{3}\right)^{\frac{2}{3}}\frac{\bar{\delta}_0}{(\theta-\sin\theta)^{\frac{2}{3}}}$$

$$\underbrace{(1+z) = (t/t_0)^{-2/3}}_{(t/t_i)^{-2/3} = \frac{(1+z_i)}{(1+z)}}$$

- I use the value I derived for A and B in the case of small perturbations.
- Note the definition of δ_0 which is the contrast at the present time.
- The equation show how the perturbation are developing as a funcion of the cosmic time.
- We would like to know, however, an estimate of when the growth of the perturbations make it necessary to pass from the linear regime to the non linear regime.

Again a summary

See next slides for details:

$$r = A(1-\cos\theta)$$

$$t = B(\theta - \sin\theta) \;\; ; \;\; A^3 = G\mathfrak{M}B^2$$

$$\bar{\delta}(r,t) = \frac{\bar{\rho}(r,t)}{\rho_b(t)} - 1 = \frac{9}{2}\frac{(\theta-\sin\theta)^2}{(1-\cos\theta)^3} - 1$$

$$\bar{\rho}(t) = \rho_b(t)\frac{9}{2}\frac{(\theta-\sin\theta)^2}{(1-\cos\theta)^3}$$

$$\bar{\delta}_0 = \frac{a_0}{a(t_i)}\frac{3\delta_i}{5} = \frac{3\delta_i}{5}(1+z_i)$$

$$r(t) = \frac{3x}{10\bar{\delta}_0}(1-\cos\theta)$$

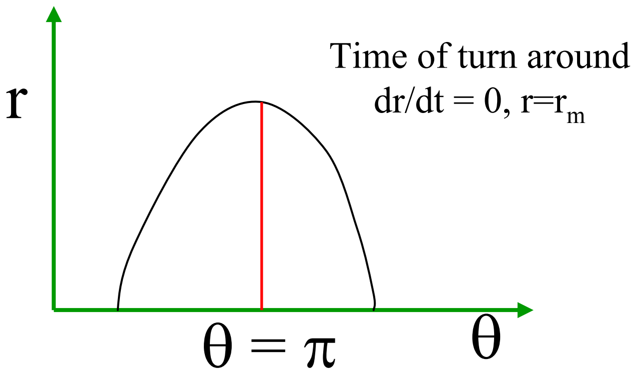
$$t = \left(\frac{3}{5}\right)^{\frac{3}{2}}\frac{3}{4}\frac{t_0}{\bar{\delta}_0^{3/2}}(\theta-\sin\theta)$$

$\Omega=1$

The easy case

And therefore I can also write

$$(1+z) = \left(t/t_0\right)^{-2/3} = \frac{5}{3}\left(\frac{4}{3}\right)^{\frac{2}{3}}\frac{\bar{\delta}}{(\theta-\sin\theta)^{\frac{2}{3}}}$$



Linear – Non Linear regime

At some point I will pass form the linear regime to the non linear regime.

I will assume, see later, that the transition from the linear regime to the non linear regime occurs when the computed growth between the two regimes differs considerably. By definition I will call that the transition moment.

In the following we summarize once again the formulae and compute the amplitude of the prturbation.

We will see that the difference is strong for $\theta = 2/3\ \pi$ when $\delta \sim 1$. That is, as roughly expected, when the extra-density in the perturbation is of the order of the background density.

$$\bar{\delta}_{linear} = \frac{3}{20}\left(\frac{6t}{B}\right)^{\frac{2}{3}} = \frac{3\delta_i}{5}\left(\frac{t}{t_i}\right)^{\frac{2}{3}} \propto a(t) \propto \frac{1}{1+z}$$

That is 3/5 of the perturbation grows as $t^{2/3}$ and for $\Omega = 1$ I can also write:

$$\frac{3\delta_i}{5}\left(\frac{t}{t_i}\right)^{\frac{2}{3}} = \frac{3}{5}\delta_i(1+z_i) = \delta_o; \textit{and in general}$$

δ_o is the present value of the density contrast as predicted by the linear theory if the density contrast at z_i was δ_i

$$\bar{\delta} = \frac{a(t)}{a(t_i)}\frac{3}{5}\delta_i = \frac{3}{5}\delta_i\frac{1+z_i}{1+z}$$

And in units of $a(t_o)$ I can write $r_i = a_i/a_o\, x = x/(1+z_i)$. That is $r_o = x$

When does the Non Linear Regime begins?

$$\bar{\delta}_{Linear}(r,t) = \frac{\bar{\rho}_{Linear}}{\rho_b} - 1 \simeq \frac{3}{5}\bar{\delta}_i \left(\frac{t}{t_i}\right)^{\frac{2}{3}} \simeq \frac{3}{5}\bar{\delta}_i \frac{(1+z_i)}{(1+z)}; \quad (1+z) = \{Slide\,17\} \frac{5}{3}\left(\frac{4}{3}\right)^{\frac{2}{3}} \frac{\bar{\delta}_0}{(\theta - \sin\theta)^{\frac{2}{3}}}$$

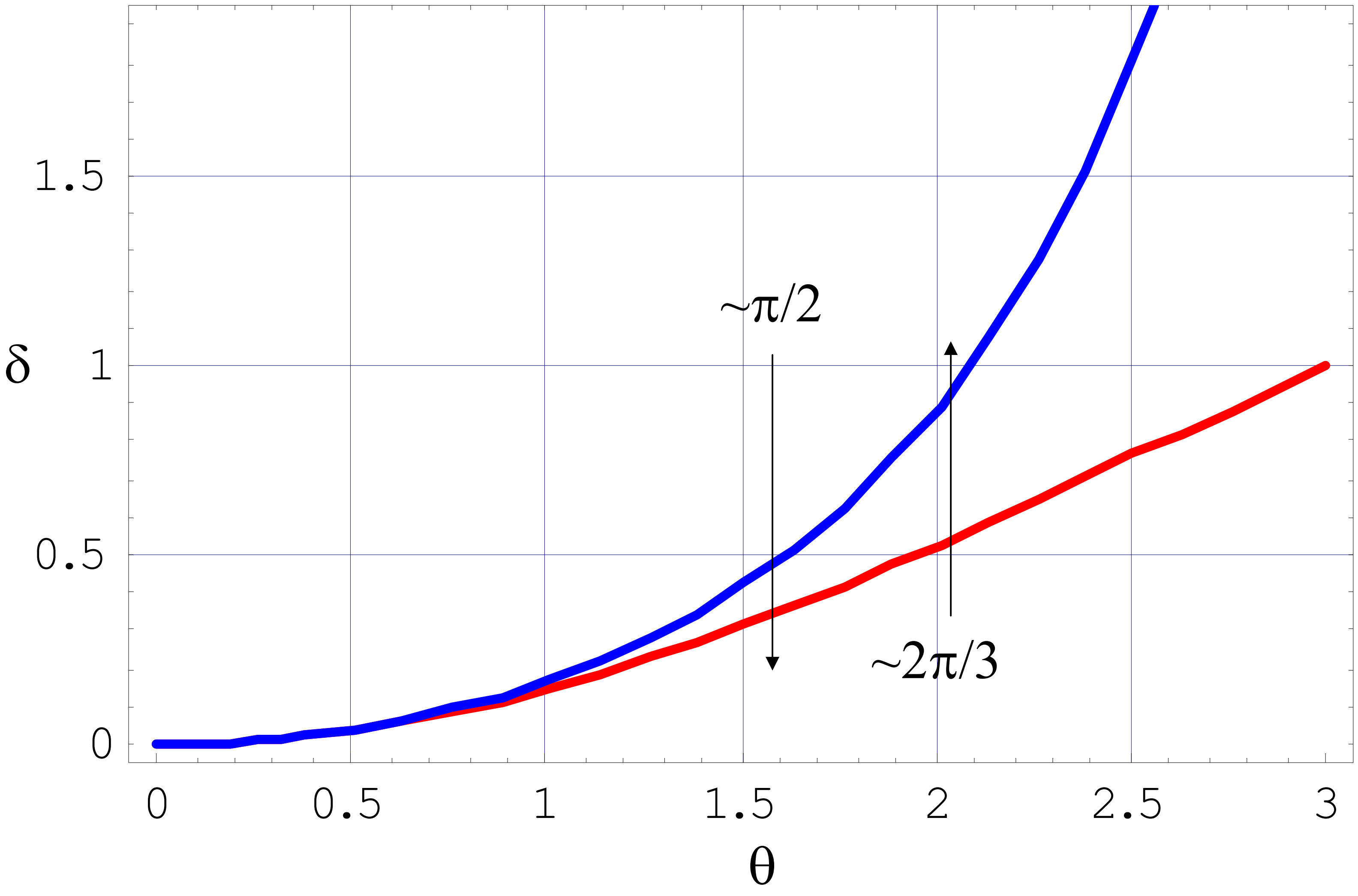
$$\bar{\delta}_{Linear}(r,t) = \frac{3}{5}\bar{\delta}_i (1+z_i) \frac{3}{5}\left(\frac{3}{4}\right)^{\frac{2}{3}} \frac{(\theta-\sin\theta)^{\frac{2}{3}}}{\bar{\delta}_0} = \frac{3}{5}\bar{\delta}_i (1+z_i) \frac{3}{5}\left(\frac{3}{4}\right)^{\frac{2}{3}} \frac{(\theta-\sin\theta)^{\frac{2}{3}}}{\frac{3}{5}\bar{\delta}_i (1+z_i)} = \frac{3}{5}\left(\frac{3}{4}\right)^{\frac{2}{3}} (\theta-\sin\theta)^{\frac{2}{3}}$$

$$z \gg 1 \quad \theta \ll 1 \quad Beginning\ of\ growth \quad \delta_{Non\ Linear}(z) = \delta_{Linear}(z)$$

$$\theta = \frac{\pi}{2} \quad \begin{cases} \delta_{Linear}(z) = \frac{3}{5}\left(\frac{3}{4}\right)^{\frac{2}{3}} (\frac{\pi}{2}-1)^{\frac{2}{3}} = 0.341 \\ \delta_{Non\ Linear}(z) = \frac{9}{2}(\frac{\pi}{2}-1)^2 - 1 = 0.466 \end{cases}$$

$$\theta = \frac{2\pi}{3} \begin{cases} \delta_{Linear}(z) = .568 \\ \delta_{Non\ Linear}(z) = 1.01 \end{cases}$$

We define the transition between the linear and the non linear regime when we reach a contrast density of about $\delta = 1$. The above computation shows that at this time the two solution differ considerably from each other.



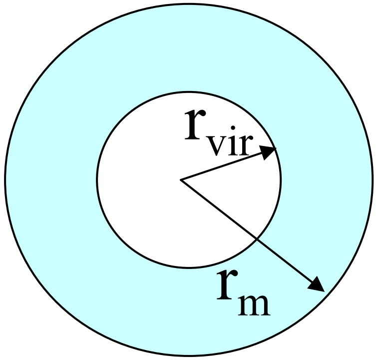
1.5
1
0.5
0
δ
~π/2
~2π/3
0
0.5
1
1.5
2
2.5
3
θ

What I would like to know:

- *At what z do I have the transition between linear and non linear?*
- *What is the ratio of the densities between perturbation and Background?*
- *At which z do we have the Maximum expansion?*
- *How large a radius do we reach? And how dense?*
- *At which redshift do I have the maximum expansion?*
- *At which redshift does the perturbation collapse?*
- *And what about Equilibrium (Virialization) and Virial parameters?*
- *What is the role of the barionic matters in all this?*

Toward Virialization

The student could also read the excellent paper by Lynden Bell on Violent Relaxation



$$At\ virialization \Rightarrow \left.\begin{array}{c} E = U + K \\ 2K + U = 0 \end{array}\right\} E = -K$$

$$at\ t = t_m \quad K = 0 \quad E = U_{Sphere} = -\frac{3}{5} G \frac{\mathfrak{M}^2}{r_m}$$

$$K = \frac{\mathfrak{M} v^2}{2} = -E = \frac{3}{5} G \frac{\mathfrak{M}^2}{r_m}$$

$$|U|[when\ virialized] = -\frac{3}{5} G \frac{\mathfrak{M}^2}{r_{vir}} = 2\ K = \mathfrak{M}\ v^2 = \frac{6}{5} G \frac{\mathfrak{M}^2}{r_m}$$

$$v = \sqrt{\left(\frac{6}{5} G \frac{\mathfrak{M}}{r_m}\right)} \qquad r_{vir} = \frac{1}{2} r_m$$

What is the density of the collapsed object?

$$\frac{\rho_{collapsed\ object\,(vir)}}{\rho_{maximum\ expansion}} = \frac{\mathfrak{M}}{\frac{4}{3}\pi\, r_{vir}^3}\,\frac{\frac{4}{3}\pi\, r_m^3}{\mathfrak{M}} = \left(\frac{r_m}{r_{ir}}\right)^3 = 8$$

$$\bar{\rho}_m(t_m) = \rho_b(t_m \to \theta = \pi)\frac{9}{2}\frac{(\theta - \sin\theta)^2}{(1-\cos\theta)^3} = \frac{9\,\pi^2}{16}\rho_b(t_m) = 5.55\,\rho_b(t_m)$$

$$\rho_b(t_m) = \frac{(1+z_m)^3}{(1+z_{collapse})^3}\rho_b(t_{collapse})$$

$$\rho_{coll} = 8\,\rho_m = 44.4\,\rho_b(t_m) = 44.4\,\frac{(1+z_m)^3}{(1+z_{collapse})^3}\rho_b(t_{collapse}) =$$

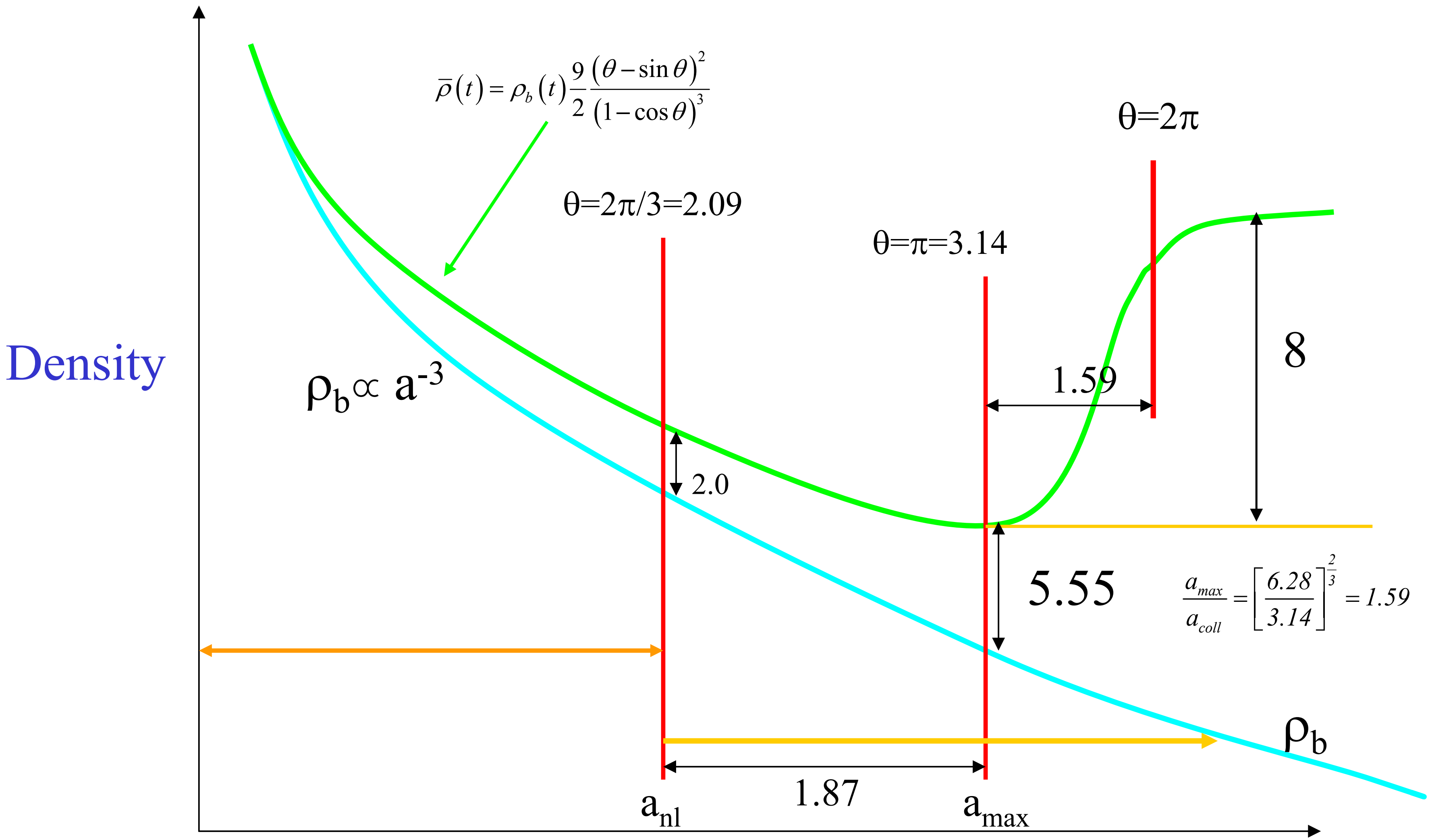
$$44.4\left(\frac{1}{0.63}\right)^3\rho_b(t_{collapse}) = 177.6\,\rho(t_0)\,(1+z_{collapse})^3$$

$$\frac{r_m}{r_{vir}} = 2 \quad ; \quad \frac{\rho_m}{\rho_b} = 1 + \delta_m = \frac{9\pi^2}{16} = 5.6$$

$$\frac{\rho_{vir}}{\rho_m} = \left(\frac{r_m}{r_{vir}}\right)^3 = 8 \quad ; \quad \frac{(1+z_{coll})}{(1+z_m)} = 0.63$$

$$\rho_{coll} = 8\rho_m = 44.4\rho_b(t_m) = 44.4\frac{(1+z_m)^3}{(1+z_{coll})^3}\rho_b(t_{coll}) =$$

$$= 177.6\rho_b(t_{coll}) = 177.6\rho_0\left(1+z_{coll}\right)^3$$



$$\frac{a_{nl}}{a_{max}} = \frac{(1+z_{max})}{(1+z_{nl})} = \left[\frac{(\theta-\sin\theta)_{\theta=2\pi/3}}{(\theta-\sin\theta)_{\theta=\pi}}\right]^{\frac{2}{3}} = \left[\frac{1.23}{3.14}\right]^{\frac{2}{3}} = 1.87$$

What happens to the baryons?

- The Dark Matter, about a facto 10 more massive, tend to attract baryons and make them follow the same evolution. However:
- During the collapse the gas involved develops shocks and heating. This generates pressure and at some point the collapse will stop.
- The agglomerate works toward equilibrium and the thermal energy must equal the gravitational energy.
- And for a mixture of Hydrogen and Helium we have:

$$\frac{3}{2}kT = \frac{1}{2}\mu m_p v^2 \qquad v^2 = \left(\frac{3G\mathfrak{M}}{5r_{vir}}\right)$$

$$\mu(Y = 0.25) = \frac{m_H n_H + m_{He} n_{He}}{2n_H + 3n_{He}} = 0.57 m_H$$

Derivation

$$\mu(Y=0.25)=\frac{m_H n_H + m_{He} n_{He}}{2n_H + 3n_{He}} = \frac{m_H}{2}\left(\frac{1+Y}{1+\frac{3n_{He}}{2n_H}}\right) =$$

$$\left.\begin{aligned} Y &= \frac{m_{He} n_{He}}{m_H n_H} = 0.25 \\ \frac{3n_{He}}{2n_H} &= \frac{3\,Y\,\frac{m_H n_H}{m_{He}}}{2n_H} = \frac{3}{2}Y\frac{1}{4} = 0.375\,Y \end{aligned}\right\} \mu(Y=0.25) = \frac{m_H}{2}\left(\frac{1+Y}{1+0.375\,Y}\right) \simeq 0.57\,m_H$$

And from Cosmology we have:

$$\rho_0 = 1.88\,10^{-29}\,\Omega h^2\, g\,cm^{-3}$$

$$t_0 \;= 0.65\,10^{10}\, h^{-1}\, yr$$

$$\delta_0 \;= 1.686\,(1 + z_{coll})$$

$$x = r_i \frac{a_0}{a(t_i)} \qquad\qquad x(t_i = t_0) = r_0$$

$$x(\mathfrak{M} = 10^{12}) = r_0 = \left[\frac{\mathfrak{M}}{\frac{4}{3}\pi\rho_0}\right]^{\frac{1}{3}} = 0.95(\Omega h^2)^{-\frac{1}{3}}\mathfrak{M}_{12}^{\frac{1}{3}} Mpc$$

Ω=1, h=1 eventually

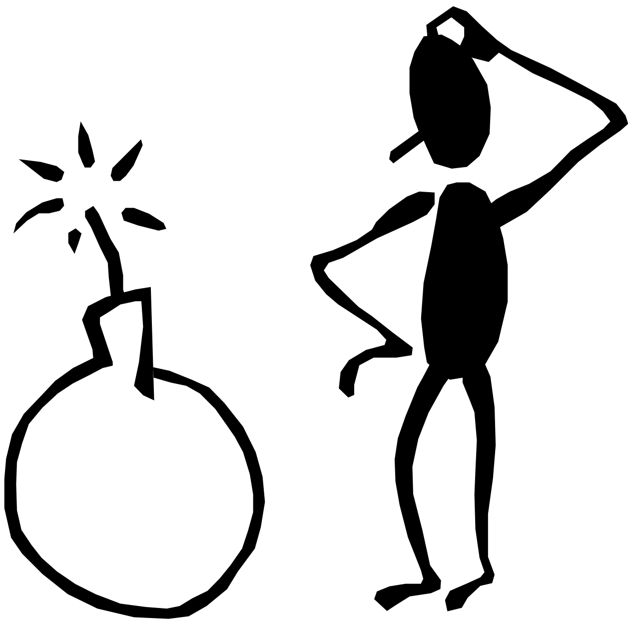
$$r_{vir} = \frac{1}{2} r_m = \frac{3}{10} \frac{x}{1.686\,(1+z_{coll})} = 169\ \ (1+z_{coll})^{-1} \mathfrak{M}_{12}^{\frac{1}{3}} h^{-\frac{2}{3}} kpc$$

or

$$r_{vir} = \left(\frac{\mathfrak{M}}{{}^4\!/\!{}_3\, \pi\, \rho_{coll}} \right)^{\frac{1}{3}} = \frac{\left(\mathfrak{M}_{12}\, 10^{12}\, 1.99\, 10^{33}\right)^{\frac{1}{3}} \Big/ \left(3.09\, 10^{21}\right)}{\left(\frac{4}{3} \pi\, 177.6\, 1.88\, 10^{-29}\, \Omega\, h^2\right)^{\frac{1}{3}} \left(1+z_{coll}\right)} =$$

$$169\,(1+z_{coll})^{-1} \mathfrak{M}_{12}^{\frac{1}{3}} h^{-\frac{2}{3}} kpc = 284\ \delta_0^{-1} \mathfrak{M}_{12}^{\frac{1}{3}} h^{-\frac{2}{3}} kpc$$

Virial velocity and Temperature



$$v = \left({3G\mathfrak{M}}/{5r_{vir}} \right)^{\frac{1}{2}} = \left({6G\mathfrak{M}}/{5r_m} \right)^{\frac{1}{2}} =$$

$$= \left[\frac{3 \bullet 6.67 \bullet 10^{-8} \bullet \mathfrak{M}_{12} 10^{12} \bullet 1.99 \bullet 10^{33}}{5 \bullet 169 \bullet \left(1 + z_{coll}\right)^{-1} \bullet \mathfrak{M}_{12}^{\frac{1}{3}} \bullet h^{-\frac{2}{3}} \bullet 3.09 \bullet 10^{21}} \right]^{\frac{1}{2}} =$$

$$123.5\ (1 + z_{coll})^{\frac{1}{2}} \mathfrak{M}_{12} h^{\frac{1}{3}}\ km/s$$

$$v = 95\ \delta_0\ \mathfrak{M}_{12}\ h^{\frac{1}{3}}\ km/s$$

$$T_{vir} = \mu\, m_H \frac{v^2}{3\,k} = \frac{0.57 \bullet 1.67 \bullet 10^{-24} \bullet 123.5^2 \bullet \left(10^5\right)^2}{1.38 \bullet 10^{-16}} =$$

$$1.0\,10^6 (1 + z_{coll})\ \mathfrak{M}_{12}^{\frac{2}{3}} h^{\frac{2}{3}}\ {}^\circ K = 5.93\,10^5\,\delta_0 \mathfrak{M}_{12}^{\frac{2}{3}} h^{\frac{2}{3}}\ {}^\circ K$$

Example

Assume a typical mass of the order of the mass of the galaxy:

$M=10^{12}\ M_{\odot}$ and h=0.5

Assume also that the mass collapse at about z=5 then we have the values of the parameters as specified below. Once the object is virialized, the value of the parameters does not change except for the evolution of the object itself.

For collapse at higher z the virial radius is smaller with higher probability of shocks.

Temperature needs viscosity and heating and ? Do we have any process making galaxies to loose angular momentum?

$$r_{vir} = \frac{169}{1+5} 0.5^{-\frac{2}{3}} = 17\ kpc$$

$$t_{coll} = t_0 \left(1+z_{coll}\right)^{-\frac{3}{2}} = 0.65\ 10^{10}\ h^{-1} / \left(6\right)^{\frac{3}{2}} = \left\{h=0.5\right\} 8.8\ 10^{8}$$

$$not\ very\ much\ time\ for\ stellar\ evolution$$

$$v_{vir} = 123.5 \left(6\right)^{\frac{1}{2}} \left(0.5\right)^{\frac{1}{3}}\ km\ s^{-1} = 240\ km\ s^{-1}$$

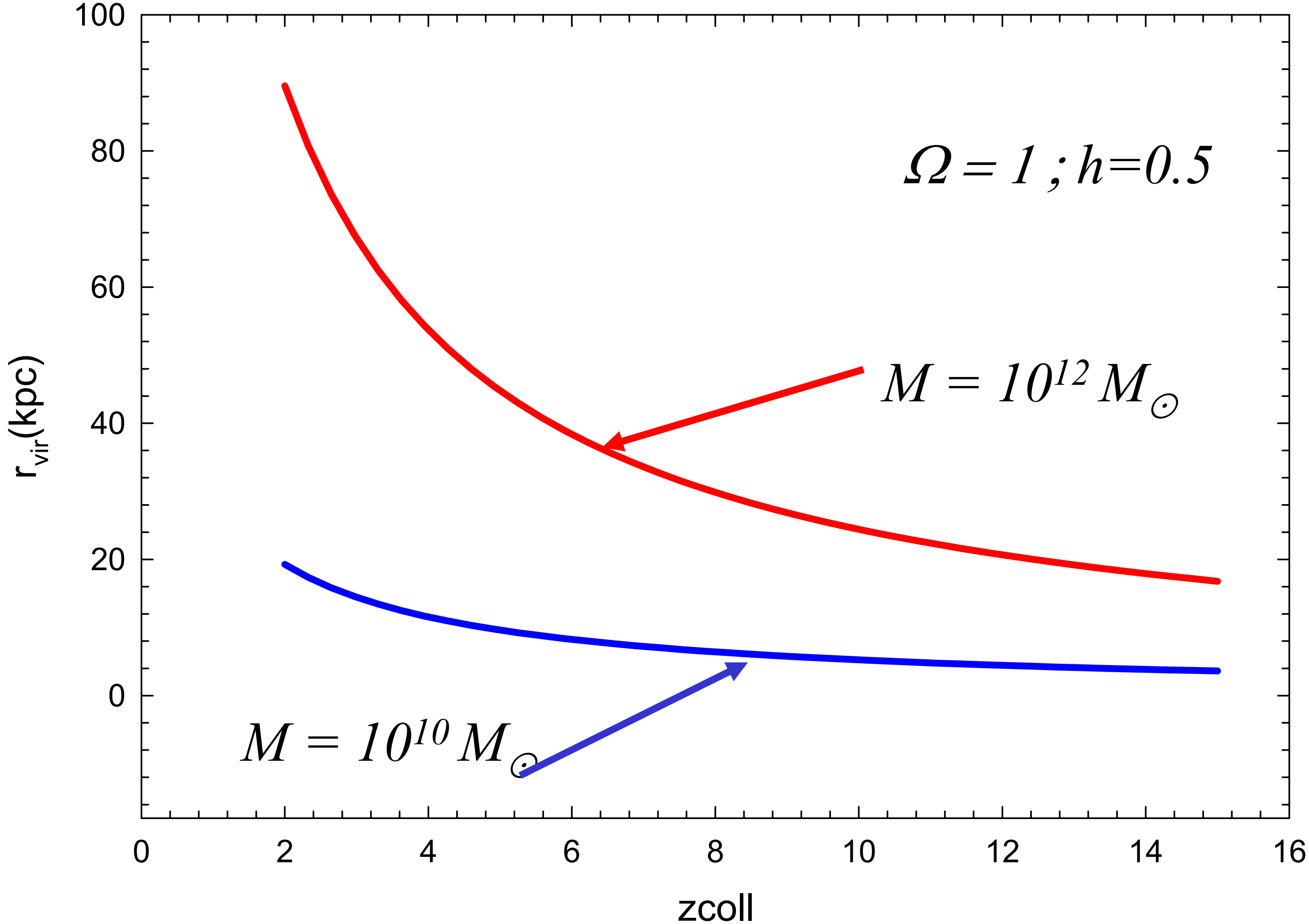
Temperature and density

$$T_{vir} \approx 10^6 \ 6 \ 0.5^{\frac{2}{3}} = 3.8 \ 10^6 \ {}^\circ K$$

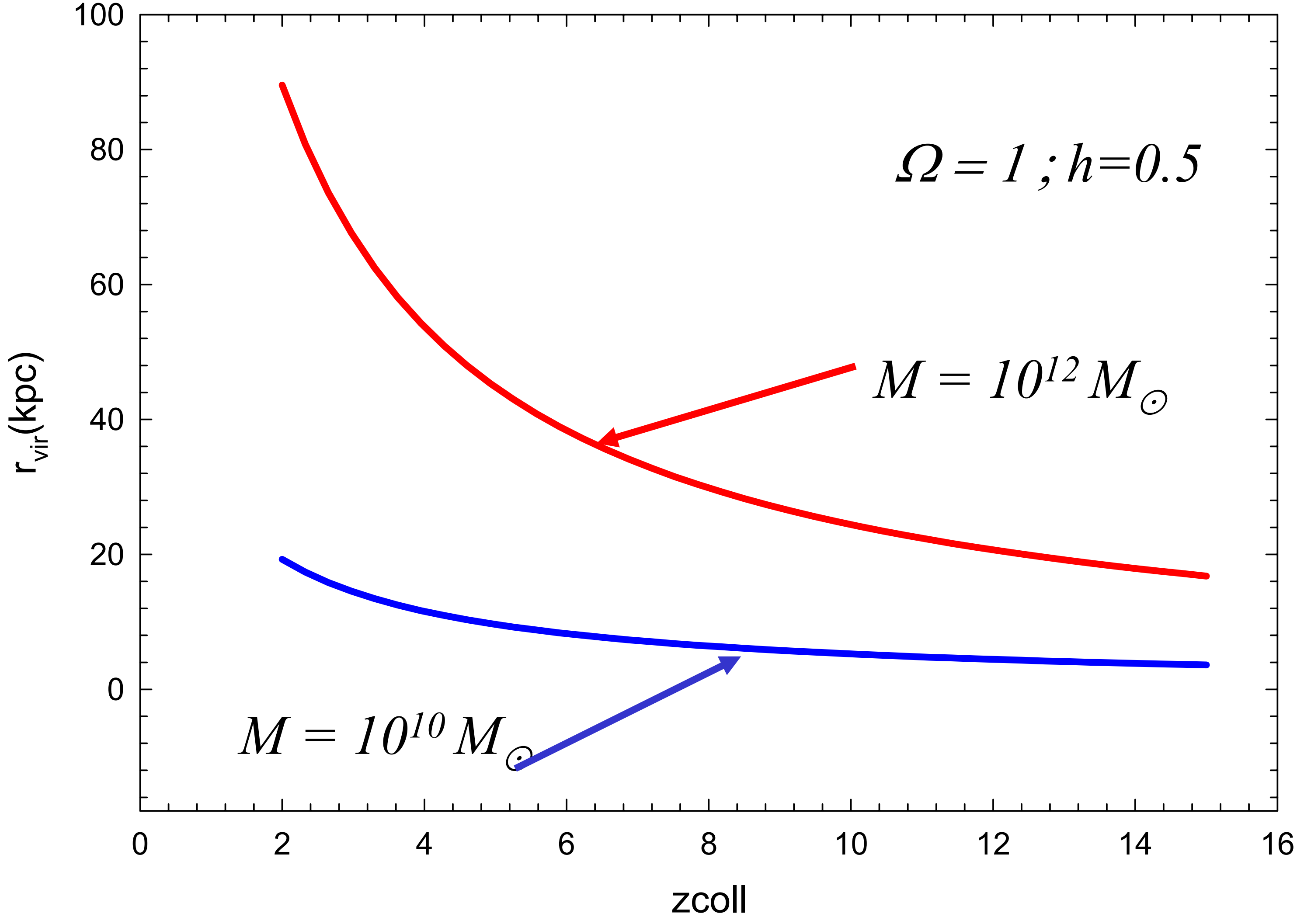
$$\frac{\rho_{coll}}{\rho_0} = 177.6\left(1+z\right)^3 \simeq 38362$$

- The temperature is very high and should emit, assuming the model is somewhat realistic, in the X ray. This however should be compared with hydro dynamical simulations to better understand what is going on.
- The density at collapse [equilibrium] is fairly high. Assuming a galaxy with a mass of about 10^{12} solar masses, a diameter of about 30 kpc and a background of 1.88 10^{-29} Ωh^2 = .9 10^{-29} the mean density would be about 5 10^4 . Very close! Coincidence?
- Obviously we should compute a density profile.

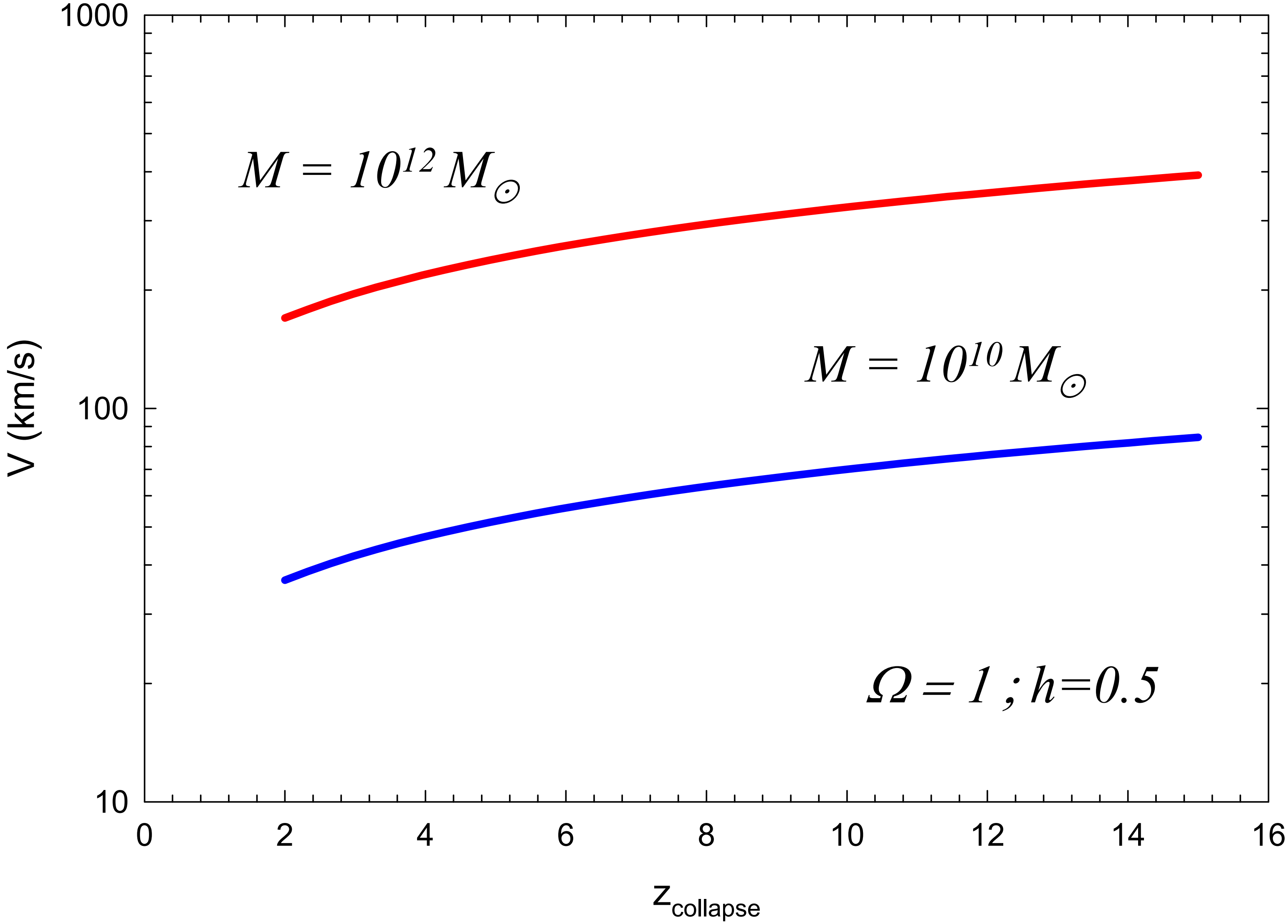
Virial Radius versus $z_{collapse}$

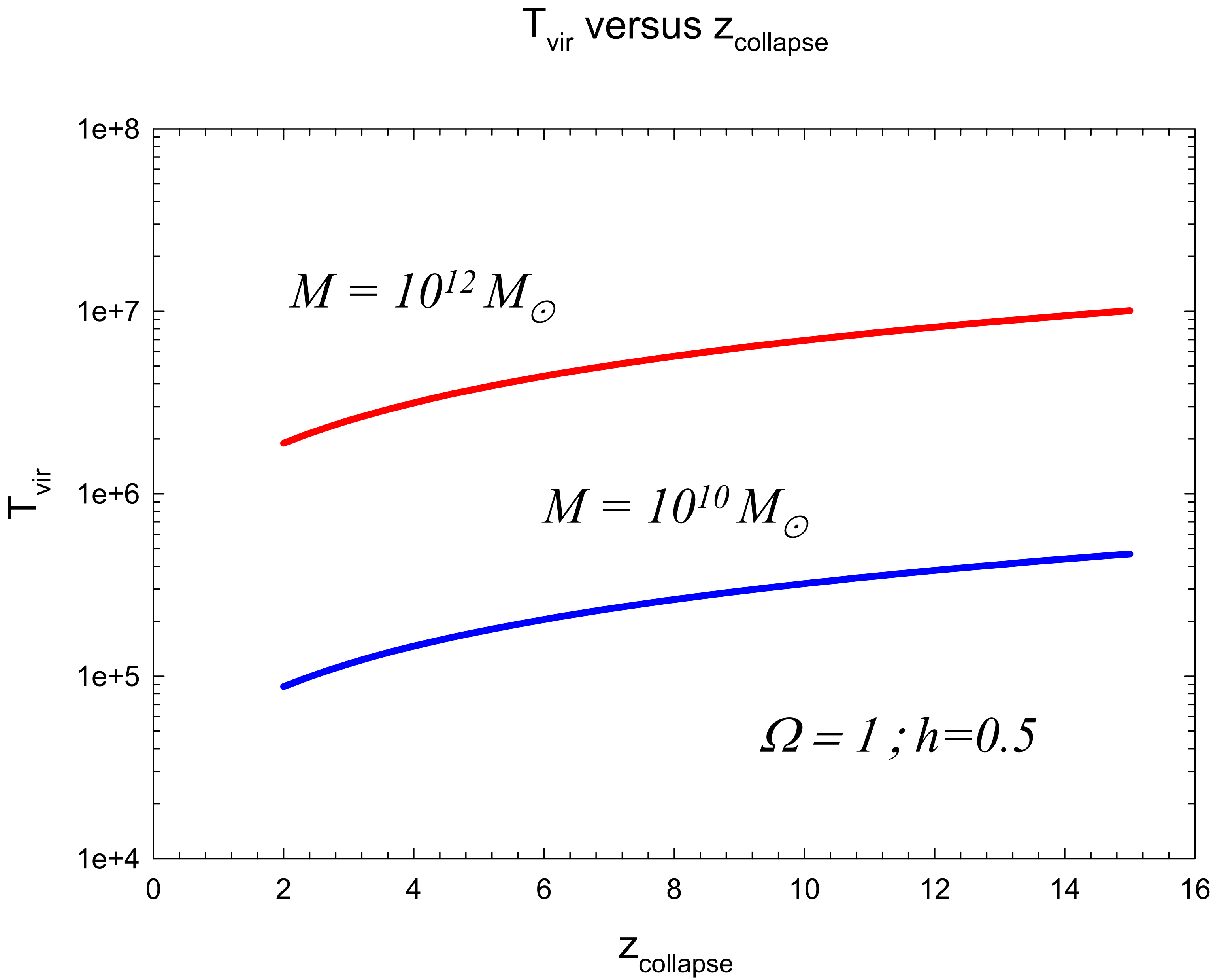


Virial Radius versus $z_{collapse}$



Virial Velocity versus $z_{collapse}$





T_vir versus z_collapse
1e+8
1e+7
1e+6
1e+5
1e+4
T_vir
0
2
4
6
8
10
12
14
16
z_collapse
M = 10^12 M_⊙
M = 10^10 M_⊙
Ω = 1 ; h=0.5