The Metric and The Dynamics

$$\Delta\tau^2 = c^2\Delta t^2 - a^2(t)\left\{\frac{\Delta r^2}{\left(1-kr^2\right)} + r^2\left(\Delta\theta^2 + Sin^2\theta\,\Delta\phi^2\right)\right\}$$

The RW metric tell us where in a 3 dimension space is an event and at which proper time. The coordinates of the event do not change as a function of time but the coordinate system will change as a function of the proper time because of the expansion factor a(t). The geometry is determined by the value of k = -1, 0, +1.

Einstein Equation – Einstein 1916

$$R_{ij} - \frac{1}{2} g_{ij} R = -\frac{8\pi G}{c^4} T_{ij}$$

Stress Tensor

$$-\Lambda_{ij}$$

Cosmological Constant

Ricci's Tensor

Ricci's scalar

In this equation figure both the geometry and the Energy

Energy momentum Tensor simplified in a Robertson Walker Metric
Metric – The Universe is homogeneous and isotropic.

$$T_{ij} = \left(p + \rho\, c^2\right) u_i u_j - p\, g_{ij}$$

The Hubble law

Separation between two observers

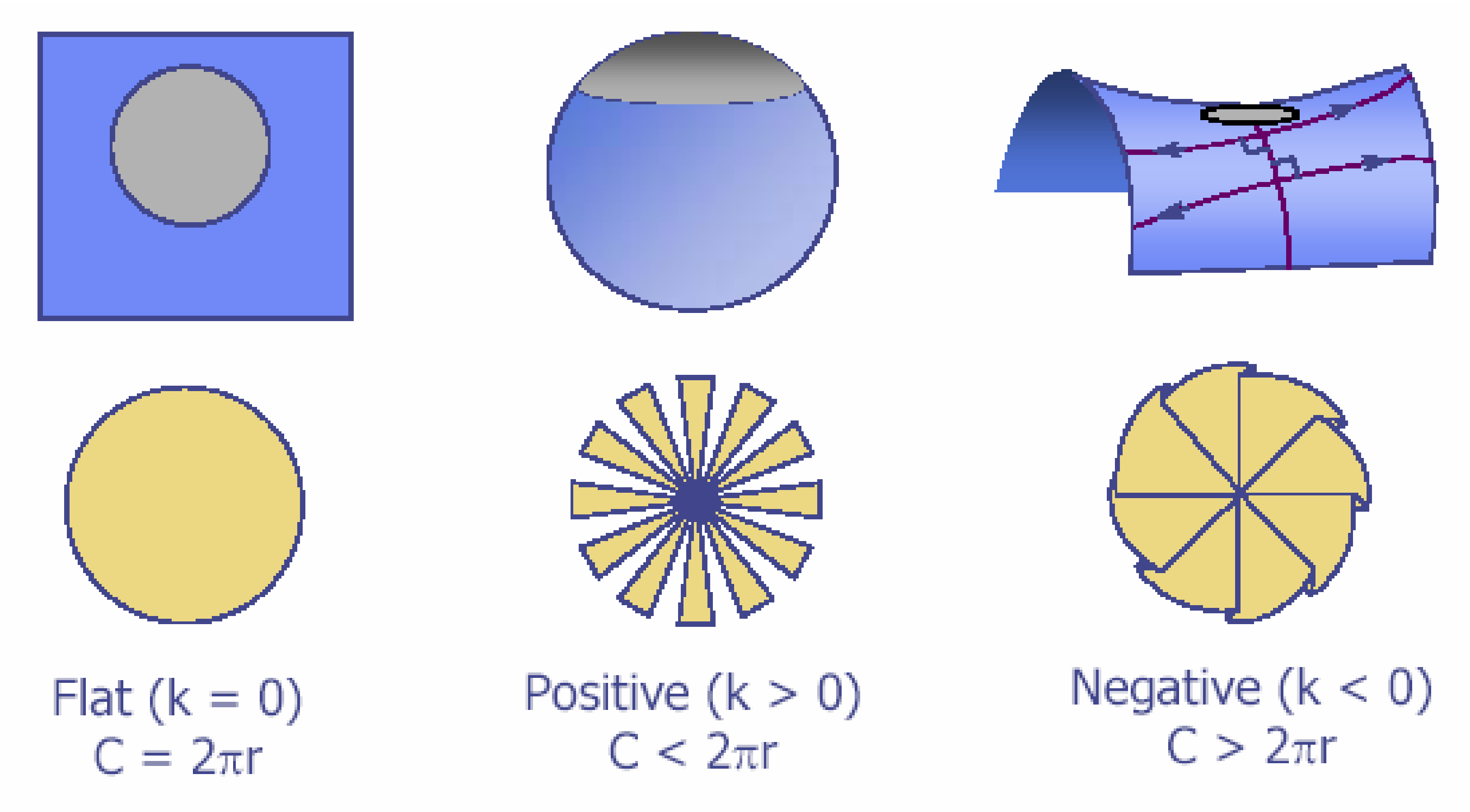
$$\delta l = a(t)\,\delta r$$

$$\frac{d}{dt}\delta l = \dot{a}(t)\,\delta r \qquad \delta r = \frac{\delta l}{a(t)}$$

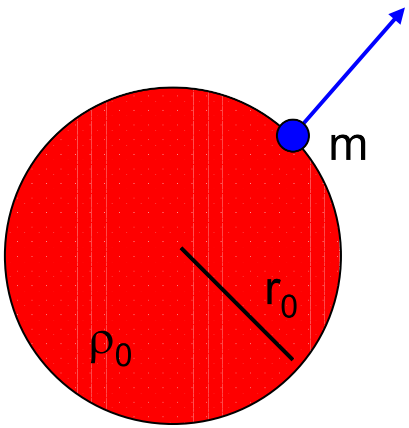
$$\frac{d}{dt}\delta l = \frac{\dot{a}(t)}{a(t)}\delta l \qquad V = H(t)\,\delta l$$

The Models and The Cosmological Parameters

Here we derive the observable as a function of different cosmological parameters. The goal is to derive a fairly complete set of equations useful for data reduction. See also the following set discussing the relation to geometry and some applications.


Flat (k = 0)
C = 2πr
Positive (k > 0)
C < 2πr
Negative (k < 0)
C > 2πr

Simple Preliminaries - Dynamics



$V=H_0 r$ We consider a sphere exdpanding with a velocity v = H r. vWe assume also that the energy and the mass are conserved

$$\frac{1}{2}mv^2 - \frac{GMm}{r} = const \equiv \begin{cases} E = 0 \\ E > 0 \\ E < 0 \end{cases}$$

Critical density

Look at it as Energy First

$$E_K = \int_0^R \frac{\rho v^2}{2} dr^3 = \frac{4\pi\rho H_0^2}{2} \int_0^R r^2 dr\, r^2 = \frac{2\pi}{5} \rho H_0^2 R^5$$

$$E_G = -G \int_0^R \frac{\mathfrak{M}(r)\rho}{r} dr^3 = -G \int_0^R \left(\frac{4}{3}\pi r^3 \rho \right) \frac{\rho}{r} dr^3 = -\frac{(4\pi)^2}{15} \rho^2 G R^5$$

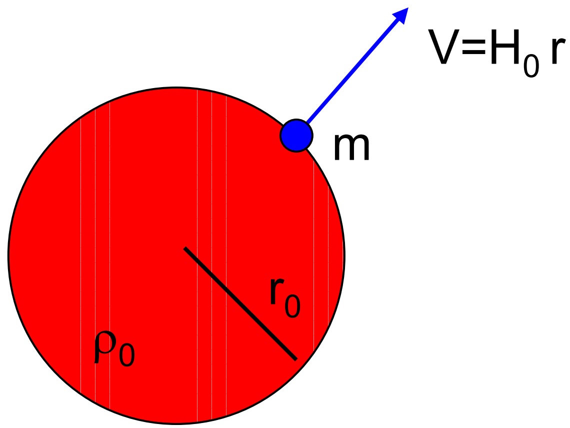
$$\frac{E_G}{E_K} = \frac{-\frac{(4\pi)^2}{15}\rho^2 G R^5}{\frac{2\pi}{5}\rho H_0^2 R^5} = -\frac{8\pi G}{3H_0^2}$$

$$E_T = E_K + E_G = E_K \left(1 + \frac{E_G}{E_K} \right) = E_K \left(1 - \rho \frac{8\pi G}{3H_0^2} \right) =$$

$$= E_K \left(1 - \frac{\rho}{\rho_c} \right) = E_K (1 - \Omega)$$

The above defines ρ_c *and* Ω

A Newtonian Model



We consider a sphere exdpanding with a velocity v = H r. vWe assume also that the energy and the mass are conserved

$$\frac{1}{2}mv^2 - \frac{GMm}{r} = const \equiv \begin{cases} E = 0 \\ E > 0 \\ E < 0 \end{cases}$$

$E = 0$

$$M = \frac{4}{3}\pi r_0^{\,3}\rho_0 \Rightarrow 0 = \frac{1}{2}H_0^2 r_0^2 - \frac{G\frac{4}{3}\pi r_0^{\,3}\rho_0}{r_0} \Rightarrow \rho_{0,c} = \frac{3H_0^2}{8\pi G}$$

Critical density

$$t = t_0 \quad \frac{1}{2}mv^2 - \frac{GMm}{r} = 0 \Rightarrow \frac{dr}{dt} = \sqrt{\frac{2GM}{r}}$$

$$\int_0^r dr\, r^{\frac{1}{2}} = \int_0^t \left(2GM\right)^{\frac{1}{2}} dt \Rightarrow t = t_0 \; ; r = r_0 \Rightarrow t_0 = \frac{\frac{2}{3} r_0^{\frac{3}{2}}}{\left(2GM\right)^{\frac{1}{2}}}$$

$$also: \frac{dr\, r^{\frac{1}{2}}}{r_0^{\frac{3}{2}}} = H_0 dt \Rightarrow solution\; r = r_0 \left(\frac{t}{t_0}\right)^{\frac{2}{3}}$$

$$\left(2GM\right)^{\frac{1}{2}} = \left(v_0^2 r_0\right)^{\frac{1}{2}} = \left(H_0^2 r_0^2 r_0\right)^{\frac{1}{2}}$$

$$t_0 = \frac{\frac{2}{3} r_0^{\frac{3}{2}}}{H_0^2 r_0^{\frac{3}{2}}} = \frac{2}{3} H_0^{-1} \quad \rho_c \simeq 2\; 10^{-29} h_0^{\,2} g\; cm^{-3}$$

$E \neq 0$

$$E = m\left(\frac{H^2}{2} - \frac{4\pi G\rho}{3}\right)r^2 = m\left(\frac{H_0^{\ 2}}{2} - \frac{4\pi G\rho_0}{3}\right)r_0^{\ 2} \ \ Divide\ by\ \frac{H_0^{\ 2}r_0^2}{2m}$$

$$= \left(\frac{H^2}{2}\frac{2}{H_0^{\ 2}r_0^2} - \frac{4\pi G\rho}{3}\frac{2}{H_0^{\ 2}r_0^2}\right)r^2 = \left(\frac{H_0^{\ 2}}{2}\frac{2}{H_0^{\ 2}r_0^2} - \frac{4\pi G\rho_0}{3}\frac{2}{H_0^{\ 2}r_0^2}\right)r_0^2$$

$$\rho_{0,c} = \frac{3H_0^2}{8\pi G}\ ;\ \Omega_0 = \frac{\rho_0}{\rho_{0,c}}$$

$$\frac{H^2r^2}{H_0^{\ 2}r_0^2} - \frac{8\pi G\rho r^2}{3H_0^{\ 2}r_0^2} = 1 - \Omega_0$$ *I conserve the mass and transformation*

$$\frac{\rho}{\rho_0} = \frac{r_0^{\ 3}}{r^3} \quad D^* = \frac{r}{r_0} \quad \tau^* = H_0 t \quad Hr = \frac{dr}{dt} \quad \frac{Hr}{H_0 r_0} = \frac{d\left(\frac{r}{r_0}\right)}{d\left(H_0 t\right)} = \frac{dD^*}{d\tau^*}$$

And I can write:

$$\left(\frac{dD^*}{d\tau^*}\right)^2 - \frac{8\pi G}{3H_0^2}\rho_0 \frac{r_0^3 r^2}{r_0^2 r^3} = 1 - \Omega_0 \quad if\ E \neq 0 \Rightarrow \Omega_0 \neq 1$$

$$\left(\frac{dD^*}{d\tau^*}\right)^2 - \frac{\Omega_0}{D^*} = 1 - \Omega_0 \quad Change\ variables\ again$$

$$\xi = \frac{|1-\Omega_0|}{\Omega_0} D^* \quad \tau = \frac{|1-\Omega_0|^{\frac{3}{2}}}{\Omega_0}\tau^* \quad dD^* = \frac{\Omega_0}{|1-\Omega_0|} d\xi \quad d\tau^* = \frac{\Omega_0}{|1-\Omega_0|^{\frac{3}{2}}} d\tau$$

$$\left(\frac{d\xi}{d\tau}\right)^2 |1-\Omega_0| - \frac{\Omega_0 |1-\Omega_0|}{\xi\Omega_0} = 1 - \Omega_0$$

$$\left(\frac{d\xi}{d\tau}\right)^2 - \frac{1}{\xi} = \begin{cases} +1 \ for\ \Omega < 1 \\ -1 \ for\ \Omega > 1 \end{cases}$$

The Equation to be solved

$$\frac{d\xi}{dt} = \left(\frac{1}{\xi} \pm 1\right)^{\frac{1}{2}} = \left(\frac{1 \pm \xi}{\xi}\right)^{\frac{1}{2}}$$

$$\int_0^{\tau} d\tau = \int_0^{\xi} \left(\frac{\xi}{1 \pm \xi}\right)^{\frac{1}{2}} d\xi$$

$$\tau \Rightarrow \tau^* \Rightarrow t \qquad \xi \Rightarrow D^* \Rightarrow r$$

$\Omega_0 > 1 \quad E<0$

$$\tau = \int_0^{\xi} \left(\frac{\xi}{1-\xi} \right)^{\frac{1}{2}} d\xi = \left\{ \xi = sin^2\left(\frac{\eta}{2}\right) \right\} = \int_0^{\eta} \left(\frac{sin^2 \frac{\eta}{2}}{cos^2 \frac{\eta}{2}} \right)^{\frac{1}{2}} 2\ sin\frac{\eta}{2} cos\frac{\eta}{2} \frac{1}{2} d\eta =$$

$$\int_0^{\eta} \frac{sin\frac{\eta}{2}}{cos\frac{\eta}{2}} sin\frac{\eta}{2} cos\frac{\eta}{2} d\eta = \int_0^{\eta} sin^2\frac{\eta}{2} d\eta = \int_0^{\eta} \frac{(1-cos\,\eta)}{2} =$$

$$since \quad sin\frac{\alpha}{2} = \sqrt{\frac{1-cos\,\alpha}{2}}$$

$$\tau = \frac{1}{2}\Big|_0^{\eta} (\eta - sin\,\eta) = \frac{1}{2}(\eta - sin\,\eta) \quad \xi = sin^2\frac{\eta}{2}$$

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$\Omega_0 < 1 \quad E>0$

$$\tau = \int_0^{\xi}\left(\frac{\xi}{1+\xi}\right)^{\frac{1}{2}} d\xi = \left\{\xi = \sinh^2\left(\frac{\eta}{2}\right)\right\} = \int_0^{\eta}\left(\frac{\sinh^2\frac{\eta}{2}}{\cosh^2\frac{\eta}{2}}\right)^{\frac{1}{2}} 2\sinh\frac{\eta}{2}\cosh\frac{\eta}{2}\frac{1}{2}d\eta =$$

$$\int_0^{\eta}\frac{\sinh\frac{\eta}{2}}{\cosh\frac{\eta}{2}}\sinh\frac{\eta}{2}\cosh\frac{\eta}{2}d\eta = \int_0^{\eta}\sinh^2\frac{\eta}{2}d\eta = \int_0^{\eta}\frac{(-1+\cosh\eta)}{2} =$$

$$since \quad \sinh\frac{\alpha}{2} = \sqrt{\frac{-1+\cosh\alpha}{2}}$$

$$\tau = \frac{1}{2}\Big|_0^{\eta}(-\eta+\sinh\eta) = \frac{1}{2}(\sinh\eta-\eta) \quad \xi = \sinh^2\frac{\eta}{2}$$

Obviously (as an example Ω>1):

$$\tau = \frac{1}{2}(\eta - \sin\eta) = \frac{|1-\Omega_0|^{\frac{3}{2}}}{\Omega_0}\tau^* = \frac{|1-\Omega_0|^{\frac{3}{2}}}{\Omega_0}H_0 t$$

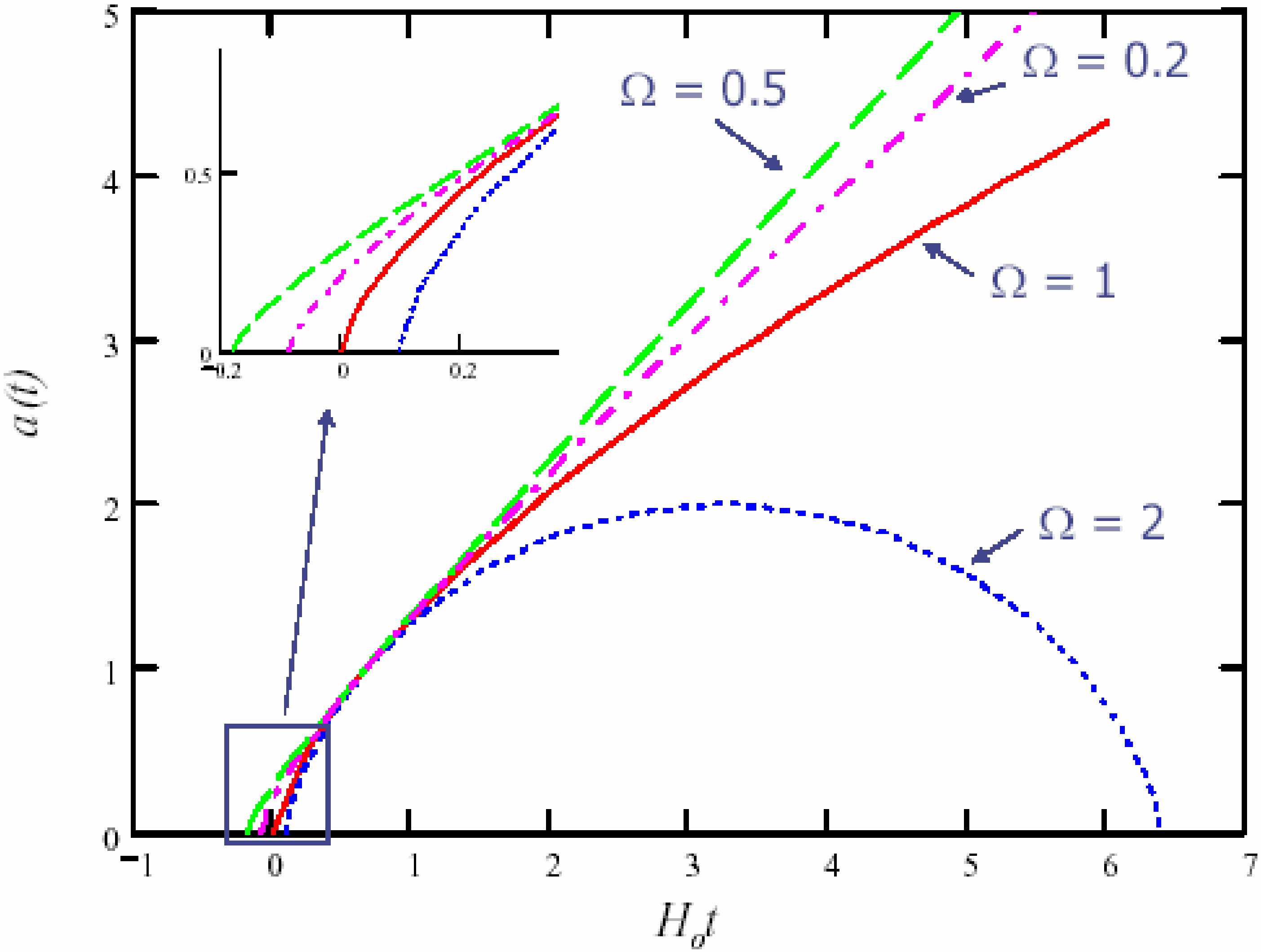
$$t = \frac{\Omega_0}{|1-\Omega_0|^{\frac{3}{2}} H_0}\frac{1}{2}(\eta - \sin\eta)$$

$$\xi = \sin^2\frac{\eta}{2} = \frac{|1-\Omega_0|}{\Omega_0}D^* = \frac{|1-\Omega_0|}{\Omega_0}\frac{r}{r_0}$$

$$\frac{r}{r_0} = \frac{\Omega_0}{|1-\Omega_0|}\sin^2\frac{\eta}{2} \; and$$

$$Maximum\ radius\ \ for\ \eta = \pi\ ;\ \xi = 1 \Rightarrow r_{max} = \frac{r_0\Omega_0}{|1-\Omega_0|} = \frac{r_0\Omega_0}{\Omega_0 - 1}$$

Plot of $\Lambda = 0$ solutions



From Einstein Equations: Friedmann 1888 - 1925

$$\frac{2\ddot{a}}{a}+\frac{\dot{a}^2+k\,c^2}{a^2}=-8\pi G\frac{p}{c^2} \quad and \quad \dot{a}^2+k\,c^2=\frac{8\pi}{3}G\rho\,a^2 \; and \; p=p(\rho)$$

$$Combined: \quad \frac{\ddot{a}}{a}=-\frac{4\pi G}{3}\left(\rho+\frac{3p}{c^2}\right)$$

This is very similar to the equation we derived in the Newtonian Toy so that the solution has been already derived for k=1, 0, -1. Here as you have seen the difference is the constant in the Equations. Furthermore I can write – see for Ω > , < , = 1:

$$\frac{k}{a^2}=\frac{1}{c^2}\left(\frac{\dot{a}}{a}\right)^2\left(\frac{\rho}{\rho_{0,c}}-1\right) \; with \; \rho_{0,c}=\frac{3}{8\pi G}\left(\frac{\dot{a}}{a}\right)^2 \; and \; \Omega=\frac{\rho}{\rho_{0,c}}$$

$$\frac{k}{a^2}=\frac{1}{c^2}\left(\frac{\dot{a}}{a}\right)^2(\Omega-1)$$

Adiabatic Expansion

The two Friedmann equations are not indipendent and are related by the adiabatic equation:

$$dE = -p\,dV$$

$$d\left(\rho\, c^2 a^3\right) = -p\, da^3$$

And in case I have $\Lambda \neq 0$ I keep the same equation using:

$$\tilde{p} = p - \frac{\Lambda c^4}{8\pi G} \; and \; \tilde{\rho} = \rho + \frac{\Lambda c^2}{8\pi G}$$

Radiation and Matter

$$d\left(\rho\, c^2 a^3\right) = -p\, da^3 \;;\; assume\ p = 0\ pressureless\ material\ dust$$

$$gas\ or\ nonrelativistic\ material \Rightarrow ideal\ gas$$

$$p = \frac{kT}{m_p c^2}\rho_m c^2 = w(T)\,\rho_m c^2\ \ with\ w(T) \ll 1$$

$$d\left(\rho\, c^2 a^3\right) = 0 \Rightarrow \rho\, a^3 = const$$

$$\rho\, a^3 = \rho_0\, a_0^3$$

$$Relativistic\ gas\ and\ Radiation$$

$$p = \frac{1}{3}\rho\, c^2$$

$$c^2\left(\rho da^3 + a^3 d\rho\right) = -\frac{1}{3}\rho c^2 da^3 \Rightarrow 4\rho a^2 da + a^3 d\rho = 0\ \ \Big|{*a}$$

$$4\rho a^3 da + a^4 d\rho = 0 \Rightarrow d\left(\rho a^4\right) = const$$

$$\rho_1\, a_1^4 = \rho_2\, a_2^4 \Rightarrow if\ \ BB \Rightarrow T(z) \propto T_0\left(1+z\right)$$

$$\dot{a}^2 + k\,c^2 = \frac{8\pi}{3} G\rho\, a^2 \Rightarrow \left(\frac{\dot{a}}{a_0}\right)^2 - \frac{8\pi G\rho}{3}\left(\frac{a}{a_0}\right)^2 = \left(for\ a = a_0\right)$$

$$H_0^2\left(1 - \frac{8\pi G\rho_0}{3H_0^{\ 2}}\right) = -\frac{kc^2}{a_0^2} \Rightarrow or \Rightarrow$$

$$H_0^2\left(1 - \frac{8\pi G\rho_0}{3H_0^{\ 2}}\right) = H_0^2\left(1 - \Omega_0\right) = -\frac{kc^2}{a_0^2}$$

$$\rho_{Tot} = \rho_R + \rho_B + \rho_{DM} + \rho_\Lambda =$$

$$= \rho_{0,c}\left[\Omega_{0,R}\left(\frac{a_0}{a}\right)^4 + \Omega_{0,B}\left(\frac{a_0}{a}\right)^3 + \Omega_{0,DM}\left(\frac{a_0}{a}\right)^3 + \Omega_\Lambda\right]$$

$$\rho_R = \rho_0\left(\frac{a_0}{a}\right)^4 = \rho_{0,c}\Omega_{0,R}\left(\frac{a_0}{a}\right)^4$$

And

$$\dot a^2 + k\,c^2 = \frac{8\pi}{3} G\rho\, a^2 \Rightarrow \left(\frac{\dot a}{a}\right)^2 - \frac{8\pi G\rho}{3}\left(\frac{a}{a}\right)^2 = -\frac{k\,c^2}{a^2}$$

$$(*\Rightarrow)\, H^2 = \frac{8\pi G\rho}{3} - \frac{k\,c^2}{a^2} = \frac{8\pi G\left(\rho_R + \rho_B + \rho_{DM} + \rho_\Lambda\right)}{3} - \frac{k\,c^2}{a^2}$$

$$H^2 = \frac{8\pi G}{3}\rho_{0,c}\left[\Omega_{0,R}\left(\frac{a_0}{a}\right)^4 + \Omega_{0,B}\left(\frac{a_0}{a}\right)^3 + \Omega_{0,DM}\left(\frac{a_0}{a}\right)^3 + \Omega_\Lambda\right] - \frac{k\,c^2}{a^2}$$

$$H^2 = H_0^2\left[\Omega_{0,R}\left(\frac{a_0}{a}\right)^4 + \Omega_{0,B}\left(\frac{a_0}{a}\right)^3 + \Omega_{0,DM}\left(\frac{a_0}{a}\right)^3 + \Omega_\Lambda\right] - \frac{k\,c^2}{a^2}\frac{H_0^2}{H_0^2}\frac{a_0^2}{a_0^2}$$

$$H^2 = H_0^2\left[\Omega_{0,R}(1+z)^4 + \Omega_{0,B}(1+z)^3 + \Omega_{0,DM}(1+z)^3 + \Omega_K(1+z)^2 + \Omega_\Lambda\right]$$

$$\text{where } \Omega_K = -\frac{kc^2}{a_0^2 H_0^2} \text{ and } \Omega_\Lambda = \frac{\Lambda\, c^2}{3H_0{}^2} \text{ from equation above } (*)$$

$$\left[\Omega_{0,R} + \Omega_{0,B} + \Omega_{0,DM} + \Omega_{0,K} + \Omega_\Lambda\right] = 1 \text{ for any } t$$

Dust Universe to make it easy

$$\left(\frac{\dot a}{a_0}\right)^2-\frac{8\pi G\rho}{3}\left(\frac{a}{a_0}\right)^2=\left(for\ a=a_0\ \&\ \rho=\rho_0\right)\Rightarrow H_0^2\left(1-\frac{8\pi G\rho_0}{3H_0^{\ 2}}\right)=H_0^2\left(1-\Omega_0\right)$$

$$\left(\frac{\dot a}{a_0}\right)^2-\frac{8\pi G\rho}{3}\left(\frac{a}{a_0}\right)^2=\left(\frac{\dot a}{a_0}\right)^2-\frac{H_0^{\ 2}}{H_0^{\ 2}}\frac{8\pi G\rho}{3}\left(\frac{a}{a_0}\right)^2=\left(\frac{\dot a}{a_0}\right)^2-H_0^{\ 2}\frac{1}{\rho_{0,c}}\rho_0\left(\frac{a_0}{a}\right)^3\left(\frac{a}{a_0}\right)^2$$

$$\left(\frac{\dot a}{a_0}\right)^2-H_0^{\ 2}\frac{\rho_0}{\rho_{0,c}}\left(\frac{a_0}{a}\right)=H_0^2\left(1-\Omega_0\right)\Rightarrow\left(\frac{\dot a}{a_0}\right)^2=H_0^2\left[\Omega_0\left(\frac{a_0}{a}\right)+\left(1-\Omega_0\right)\right]$$

$$f(r)=\int_0^r\frac{dr}{\sqrt{1-kr^2}}=\int_t^{t_0}\frac{c\,dt}{a(t)}=\int_t^{t_0}\frac{c\,dt}{a(t)}\frac{\dot a\,dt}{\dot a\,dt}=\int_a^{a_0}\frac{c\,da}{a(t)\,\dot a}$$

$$f(r)=\int_a^{a_0}\frac{c\,da}{a(t)\,\dot a}=\frac{c}{a_0H_0}\int_a^{a_0}\left[\Omega_0\left(\frac{a_0}{a}\right)+\left(1-\Omega_0\right)\right]^{-\frac{1}{2}}\frac{a_0}{a}d\left(\frac{a}{a_0}\right)$$

Integrating both

$$r=\frac{2c}{H_0a_0}\frac{\Omega_0z+\left(\Omega_0-2\right)\left[-1+\left(\Omega_0z+1\right)^{\frac{1}{2}}\right]}{\Omega_0^{\ 2}\left(1+z\right)}$$

$$\left(\frac{\dot{a}}{a_0}\right)^2 = H_0^2\left[\Omega_0\left(\frac{a_0}{a}\right) + (1-\Omega_0)\right] \quad \Leftrightarrow * \frac{a_0^2}{a^2}$$

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2\frac{a_0^2}{a^2}\left[\Omega_0\left(\frac{a_0}{a}\right) + (1-\Omega_0)\right]$$

$$H^2(z) = H_0^2(1+z)^2\left[\Omega_0(1+z) + (1-\Omega_0)\right]$$

$$\Omega(z) = \frac{\rho}{\rho_c} = \frac{\rho_0(1+z)^3\, 8\pi G}{3H^2(z)} = \frac{\rho_0(1+z)^3}{\frac{3}{8\pi G}H_0^2(1+z)^2\left[\Omega_0(1+z)+(1-\Omega_0)\right]}$$

$$\Omega(z) = \frac{\Omega_0(1+z)}{\Omega_0(1+z)+(1-\Omega_0)} = \frac{\Omega_0(1+z)}{1+\Omega_0 z} \rightarrow 1 \text{ for } z \rightarrow \infty$$

$$H^2(z) = H_0^2 (1+z)^2 \left[\Omega_0 (1+z) + (1-\Omega_0) \right]$$

$$\frac{da}{dt}\frac{1}{a} = H_0 (1+z)\sqrt{\Omega_0\ z+1}\ ;\ \frac{a}{a_0} = \frac{1}{1+z}\ ;\ da = \frac{a_0}{(1+z)^2}dz$$

$$dt = \frac{da}{a\, H_0 (1+z)\sqrt{\Omega_0\ z+1}} = \frac{a_0}{(1+z)^2\, a\, H_0 (1+z)\sqrt{\Omega_0\ z+1}}dz$$

$$dt = \frac{(1+z)}{(1+z)^2\, H_0 (1+z)\sqrt{\Omega_0\ z+1}}dz = \frac{dz}{H_0 (1+z)^2 \sqrt{\Omega_0\ z+1}}$$

$$t(z) = \frac{1}{H_0}\int_z^\infty \frac{dz}{(1+z)^2 \sqrt{\Omega_0\ z+1}} = (for\ \Omega_0 = 1) = \frac{2}{3} H_0^{-1} \frac{1}{(1+z)^{\frac{3}{2}}}$$

Angular separation

From Mag Notes: $ds^2 = -r_1^2 \; a^2(t_1)\, \Delta\theta^2 = -d^2$

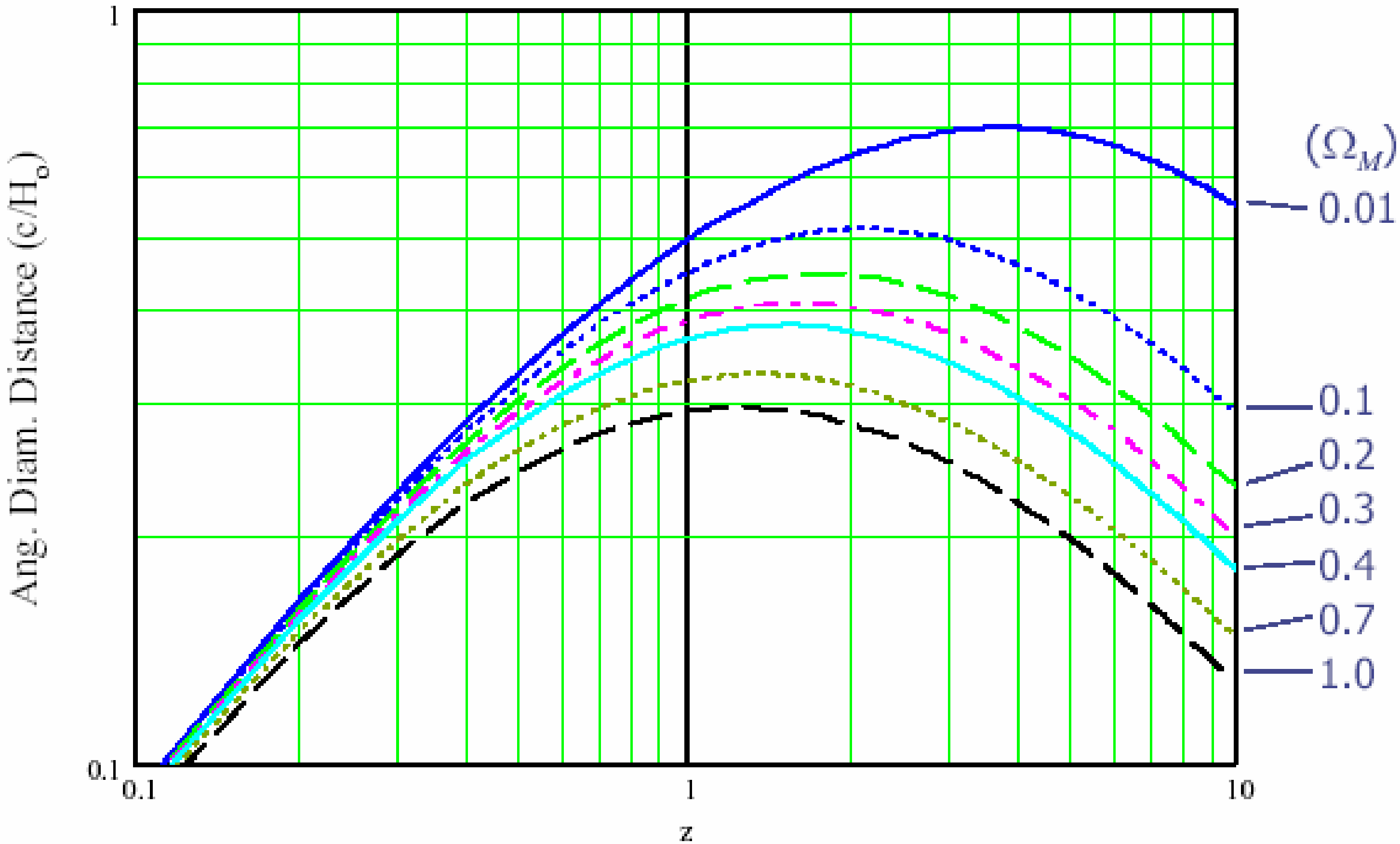
$$\Delta\theta = \frac{d}{a(t)\,r} \qquad a = a_0 \frac{1}{(1+z)}$$

$$r = \frac{2c}{H_0 a_0} \frac{\Omega_0 z + (\Omega_0 - 2)\left[-1 + (\Omega_0 z + 1)^{\frac{1}{2}}\right]}{\Omega_0^{\,2}(1+z)}$$

$$\Delta\theta = \frac{H_0 a_0}{2\,c} \frac{(1+z)}{a_0} \frac{d\ \Omega_0^{\,2}(1+z)}{\Omega_0 z + (\Omega_0 - 2)\left[-1 + (\Omega_0 z + 1)^{\frac{1}{2}}\right]} =$$

$$\frac{H_0}{2\,c} \frac{d\ \Omega_0^{\,2}(1+z)^2}{\Omega_0 z + (\Omega_0 - 2)\left[-1 + (\Omega_0 z + 1)^{\frac{1}{2}}\right]}$$

Angular diameter distance: $\Omega_M + \Omega_\Lambda = 1$ models



Integration, Mattig solution

see transparencies

- The problem is to derive the solution of the Friedmann equations. In other words we want to have r or better a_0r as a function of observable as z, Ω, H_0 …
- We assume for now $\Lambda = 0.0$

$$I\ basically\ must\ solve:$$

$$\int_0^r \frac{dr}{\sqrt{1-kr^2}} = \int_t^{t_0} \frac{c\,dt}{a}$$

$$we\ will\ find\ r = Sin(\vartheta_0 - \vartheta)\ where\ \vartheta\ is\ related\ to\ H_0\ \Omega_0\ z$$

$$and\ Sin(\vartheta_0 - \vartheta)\ to\ H_0\ \Omega_0\ and\ we\ \exp licit\ this$$

$$\left.\begin{array}{l}\ddot{a} = -\dfrac{4\pi G}{3}\left(\rho + 3\dfrac{p}{c^2}\right)a \\[2ex] a\,\ddot{a} + 2\dot{a}^2 + 2k\,c^2 = 4\pi G\left(\rho + \dfrac{p}{c^2}\right)a^2\end{array}\right\} \Rightarrow \dot{a}^2 + k\,c^2 = \frac{8\pi}{3}G\rho\,a^2$$

$$\dot{a}^2 + k\,c^2 = \frac{8\pi}{3}G\rho\,a^2 \Rightarrow \left(\frac{\dot{a}}{a_0}\right)^2 - \frac{8\pi G\rho}{3}\left(\frac{a}{a_0}\right)^2 = \left(for\ a = a_0\right)$$

$$H_0^2\left(1 - \frac{8\pi G\rho_0}{3H_0{}^2}\right) = -\frac{kc^2}{a_0^2} \Rightarrow or \Rightarrow$$

$$H_0^2\left(1 - \frac{8\pi G\rho_0}{3H_0{}^2}\right) = H_0^2\left(1 - \Omega_0\right) = -\frac{kc^2}{a_0^2}$$

For k=1

$$H_0^2\left(1-\frac{8\pi G\rho_0}{3H_0^{\,2}}\right)=H_0^2\left(1-\Omega_0\right)=-\frac{c^2}{a_0^2}\quad furthermore\ since$$

$$\rho_0=\rho_{0,c}\Omega_0\ and\ a_0^2=-\frac{c^2}{H_0^2\left(1-\Omega_0\right)}$$

$$\dot{a}^2+c^2=\frac{8\pi}{3}G\rho\, a^2=\frac{8\pi}{3}G\frac{\rho_0 a_0^3}{a}=\frac{8\pi}{3}G\Omega_0\frac{3H_0^2}{8\pi G}\frac{1}{a}\left[-\frac{c^2}{H_0^2\left(1-\Omega_0\right)}\right]^{3/2}=$$

$$=\Omega_0\frac{1}{a}\frac{c^3}{H_0}\frac{1}{\left(\Omega_0-1\right)^{3/2}}\Leftrightarrow Define\ \alpha=\frac{c}{H_0}\frac{\Omega_0}{\left(\Omega_0-1\right)^{3/2}}$$

$$\dot{a}^2=c^2\left(\frac{\alpha}{a}-1\right)$$

$$\frac{da}{dt} = c\sqrt{\frac{\alpha}{a} - 1} \Rightarrow \left(* \frac{1}{a} \right) \Rightarrow \frac{cdt}{a} = \frac{da}{a\sqrt{\frac{\alpha - a}{a}}} = \frac{da}{\sqrt{a\left(\alpha - a\right)}}$$

Change variable: $a = \alpha \, \mathrm{Sin}^2\left(\theta/2\right)$

$$da = \frac{1}{2}\alpha \sin\theta d\theta$$

$$\int_a^{a_0} \frac{da}{\sqrt{a\left(\alpha - a\right)}} = \int_\theta^{\theta_0} \frac{\frac{1}{2}\alpha \sin\theta d\theta}{\sqrt{\alpha \, \mathrm{Sin}^2\left(\theta/2\right)\left(\alpha - \alpha \, \mathrm{Sin}^2\left(\theta/2\right)\right)}} =$$

$$\int_\theta^{\theta_0} \frac{\frac{1}{2}\alpha \sin\theta d\theta}{\theta\sqrt{\alpha^2 \, \mathrm{Sin}^2\left(\theta/2\right) - \alpha^2 \, \mathrm{Sin}^4\left(\theta/2\right)}} = \int_\theta^{\theta_0} \frac{\frac{1}{2}\alpha \sin\theta d\theta}{\alpha \, \mathrm{Sin}\left(\theta/2\right)\sqrt{\left(1 - \mathrm{Sin}^2\left(\theta/2\right)\right)}} =$$

$$\int_{\theta}^{\theta_0} \frac{\frac{1}{2}\alpha \sin\theta d\theta}{\alpha\,\mathrm{Sin}\left(\theta/2\right)\sqrt{\left(1-\mathrm{Sin}^2\left(\theta/2\right)\right)}} = \int_{\theta}^{\theta_0} \frac{\frac{1}{2}\alpha \sin\theta d\theta}{\alpha\,\mathrm{Sin}\left(\theta/2\right) Cos\left(\theta/2\right)} =$$

$$\int_{\theta}^{\theta_0} \frac{\frac{1}{2}\alpha \sin\theta d\theta}{1/2\,\alpha\, Sin\,\theta} = \int_{\theta}^{\theta_0} d\theta = \theta_0 - \theta$$

Now we look at the metric's equation

$$\int_0^r \frac{dr}{\sqrt{1-kr^2}} = \int_t^{t_0} \frac{c\,dt}{a} = \left(as\ we\ have\ just\ derived\right) = \theta_0 - \theta$$

$$for\ k=1 \Rightarrow \int_0^r \frac{dr}{\sqrt{1-r^2}} = ArcSin\left(r\right)$$

$$ArcSin\left(r\right) = \theta_0 - \theta \Rightarrow r = Sin\left(\theta_0 - \theta\right)\ or$$

$$r = Sin\left(\theta_0 - \theta\right) = r = Sin\left(\theta_0\right)Cos\left(\theta\right) - Cos\left(\theta_0\right)Sin\left(\theta\right)$$

$$and$$

$$1+z = \frac{a_0}{a} = \frac{Sin^2 \theta_0/2}{Sin^2 \theta/2}$$

Now we prepare some transformations

$$Sin(\theta) =$$

$$= 2Sin\left(\theta/2\right)Cos\left(\theta/2\right) = 2Sin\left(\theta/2\right)\sqrt{1 - Sin^2\left(\theta/2\right)} =$$

$$= 2\frac{Sin\left(\theta_0/2\right)}{\sqrt{1+z}}\sqrt{1 - \frac{Sin^2\left(\theta_0/2\right)}{(1+z)}} = 2\frac{Sin\left(\theta_0/2\right)}{\sqrt{1+z}}\sqrt{\frac{Cos^2\left(\theta_0/2\right)+z}{(1+z)}} =$$

$$= 2\frac{Sin\left(\theta_0/2\right)}{1+z}\sqrt{Cos^2\left(\theta_0/2\right)+z}$$

$$Cos(\theta) =$$

$$= Cos^2\left(\theta/2\right) - Sin^2\left(\theta/2\right) = 1 - 2\,Sin^2\left(\theta/2\right) = 1 - 2\,\frac{Sin^2\left(\theta_0/2\right)}{1+z} =$$

$$\frac{1 + z - 2\,Sin^2\left(\theta/2\right)}{1+z} = \frac{Cos^2\left(\theta/2\right) - Sin^2\left(\theta/2\right) + z}{1+z} = \frac{Cos(\theta_0) + z}{1+z}$$

Now from slide 28

$$H_0^2\left(1-\Omega_0\right)=-\frac{c^2}{a_0^2}\quad ; \qquad \alpha=\frac{c}{H_0}\frac{\Omega_0}{\left(\Omega_0-1\right)^{3/2}}\ and\ I\ can\ write:$$

$$\tfrac{1}{2}\alpha\left(1-Cos\left(\theta_0\right)\right)=\tfrac{1}{2}\alpha\ Sin^2\left(\theta_0/2\right)=a_0\left(by\ definition\right)=$$

$$=\frac{c}{\sqrt{H_0^2\left(\Omega_0-1\right)}}=\frac{\left(\Omega_0-1\right)^{-1/2}\left(\Omega_0-1\right)^{3/2}\alpha}{\Omega_0}=\alpha\frac{\Omega_0-1}{\Omega_0}$$

$$Cos\left(\theta_0\right)=\left[\tfrac{1}{2}-\frac{\Omega_0-1}{\Omega_0}\right]2=\frac{\Omega_0-2\Omega_0+2}{2\Omega_0}2=\frac{2-\Omega_0}{\Omega_0}$$

$$Sin\left(\theta_0\right)=\left[1-Cos^2\left(\theta_0\right)\right]=\frac{2\sqrt{\Omega_0-1}}{\Omega_0}$$

Summary of relations

$$r = Sin\left(\theta_0\right)Cos\left(\theta\right) - Cos\left(\theta_0\right)Sin\left(\theta\right)$$

$$Sin\left(\theta_0\right) = \frac{2\sqrt{\Omega_0 - 1}}{\Omega_0}; \qquad Cos\left(\theta_0\right) = \frac{2 - \Omega_0}{\Omega_0}$$

$$Cos\left(\theta\right) = \frac{Cos\left(\theta_0\right) + z}{1 + z}; \quad Sin\left(\theta\right) = 2\frac{Sin\left(\theta_0/2\right)}{1 + z}\sqrt{Cos^2\left(\theta_0/2\right) + z}$$

Additional relations

$$Cos(\theta_0) = \frac{2-\Omega_0}{\Omega_0} = Cos^2\left(\theta_0/2\right) - Sin^2\left(\theta_0/2\right)$$

$$\frac{2-\Omega_0}{\Omega_0} = Cos^2\left(\theta_0/2\right) - 1 + Cos^2\left(\theta_0/2\right) = 2Cos^2\left(\theta_0/2\right) - 1$$

$$Cos\left(\theta_0/2\right) = \sqrt{\left(\frac{2-\Omega_0}{\Omega_0}+1\right)1/2} = \sqrt{\frac{1}{\Omega_0}}$$

$$Sin(\theta_0) = \frac{2\sqrt{\Omega_0 - 1}}{\Omega_0} = 2Sin\left(\theta_0/2\right)Cos\left(\theta_0/2\right)$$

$$Sin\left(\theta_0/2\right) = \frac{2\sqrt{\Omega_0 - 1}}{\Omega_0}\sqrt{\Omega_0}\,1/2 = \sqrt{\frac{\Omega_0 - 1}{\Omega_0}}$$

$$r = Sin(\theta_0)Cos(\theta) - Cos(\theta_0)Sin(\theta) =$$

$$= \frac{2\sqrt{\Omega_0 - 1}}{\Omega_0} \frac{Cos(\theta_0) + z}{1+z} - \frac{2-\Omega_0}{\Omega_0} 2 \frac{Sin\left(\theta_0/2\right)}{1+z} \sqrt{Cos^2\left(\theta_0/2\right) + z} =$$

$$= \frac{2\sqrt{\Omega_0 - 1}}{\Omega_0} \frac{\frac{2-\Omega_0}{\Omega_0} + z}{1+z} - \frac{2-\Omega_0}{\Omega_0} 2 \frac{\sqrt{\frac{\Omega_0 - 1}{\Omega_0}}}{1+z} \sqrt{\frac{1}{\Omega_0} + z}$$

$$r = \frac{2\sqrt{\Omega_0 - 1}}{\Omega_0^2 (1+z)} \left[\Omega_0 z + (2-\Omega_0) - (2-\Omega_0)\sqrt{\Omega_0 z + 1} \right] =$$

$$r = \frac{2\sqrt{\Omega_0 - 1}}{\Omega_0^2 (1+z)} \left[\Omega_0 z + (2-\Omega_0)\left(1 - \sqrt{\Omega_0 z + 1}\right) \right] \; and$$

$$slide\ 28 \Rightarrow H_0^2 (1-\Omega_0) = -\frac{c^2}{a_0^2}$$

$$r = \frac{2\,c}{H_0 a_0 \Omega_0^2 (1+z)} \left[\Omega_0 z + (2-\Omega_0)\left(1 - \sqrt{\Omega_0 z + 1}\right) \right] \quad or\ r\ a_0 = ...$$

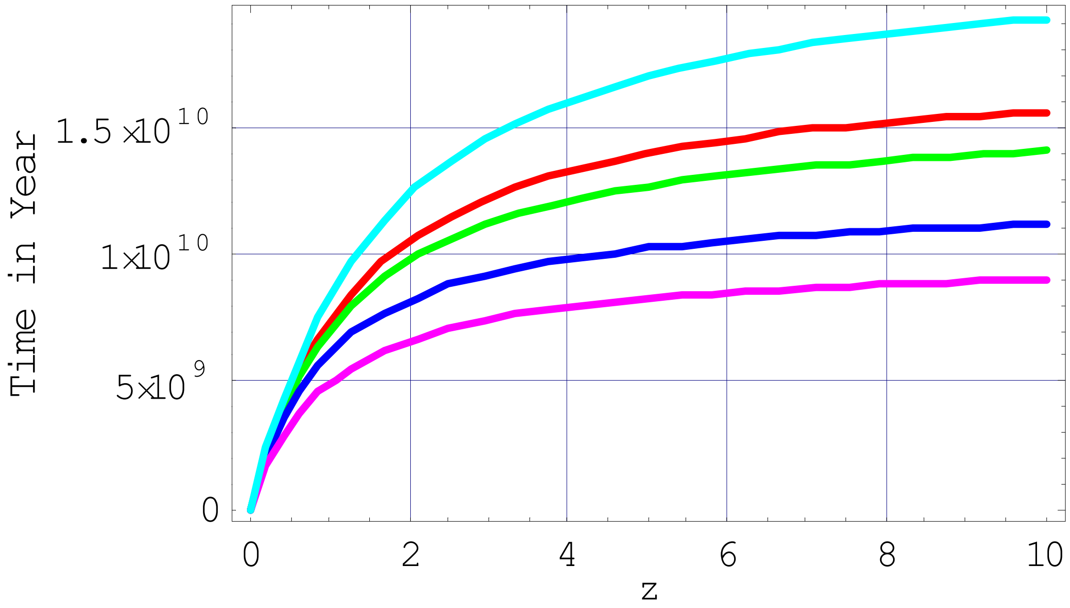
Generalities

- As we have seen we can integrate for a dust Universe. We can derive a compact equation also when we account for a Matter + Radiation Universe. It is impossible to integrate if we account also for the cosmological constant or Vacuum Energy. In this case we must integrate numerically.
- On the other hand we must also remember, as we will see shortly that the present Universe is a Dust Universe so that what is needed to the observer are the equations we are able to integrate.
- Recent results, however, both on the Hubble relation using Supernovae and on the MWB, with Boomeramg and WMAP , have shown that the cosmological constant is dominant and $\Omega_{tot}=1$ so that we should use such relations.

Compare with toy Model: Ω_m=1, Ω_Λ=0 & The real thing

$$t_1 = 0 \Rightarrow \text{The beginning } z = \infty$$

$$t_0 = \frac{1}{H_0}\int_0^\infty \frac{1}{(1+z)\sqrt{(1+z)^3}}dz = \frac{1}{H_0}\frac{2}{3} \Rightarrow 1.4\ 10^{10}\ \text{years}$$



Ω_m=0.3, Ω_Λ =0.9

Ω_m=0.3, Ω_Λ =0.7

Ω_m=0.3, Ω_Λ =0.0

Ω_m=1.0, Ω_Λ =0.0

Ω_m=0.1, Ω_Λ=0.9

The look back time

$$H^2 = H_0^2\left[\Omega_{0,R}(1+z)^4 + \Omega_{0,B}(1+z)^3 + \Omega_{0,DM}(1+z)^3 + \Omega_K(1+z)^2 + \Omega_\Lambda\right]$$

$$H = \frac{\dot{a}}{a} = a_0\frac{1}{a_0 a}\frac{d}{dt}a = \frac{d}{dt}\log\left(\frac{a(t)}{a_0}\right) = \frac{d}{dt}\log\left(\frac{1}{1+z}\right) = -\frac{1}{1+z}\frac{dz}{dt}$$

$$\frac{dt}{dz} = -\frac{1}{1+z}\frac{1}{H} = -\frac{1}{H_0(1+z)\sqrt{\left[\Omega_{0,R}(1+z)^4 + \Omega_{0,B}(1+z)^3 + \Omega_{0,DM}(1+z)^3 + \Omega_K(1+z)^2 + \Omega_\Lambda\right]}}$$

$$However\ \Omega_m = \Omega_B + \Omega_{DM}\ \ and\ \left[\Omega_{0,R} + \Omega_{0,B} + \Omega_{0,DM} + \Omega_{0,K} + \Omega_\Lambda\right] = 1$$

$$I\ also\ disregard\ radiation\ (compute\ the\ density) \Rightarrow \Omega_{0,K} = 1 - \Omega_m - \Omega_\Lambda$$

$$\frac{dt}{dz} = -\frac{1}{H_0(1+z)\sqrt{\left[\Omega_{0,m}(1+z)^3 + \left[1-\Omega_{0,m}-\Omega_\Lambda\right](1+z)^2 + \Omega_\Lambda\right]}} \Rightarrow$$

$$\frac{dt}{dz} = -\frac{1}{H_0(1+z)\sqrt{\left(\left[1+\Omega_{0,m}z\right](1+z)^2 - z\left[2+z\right]\Omega_\Lambda\right)}}\ \ accounting\ \ for\ the - sign$$

$$t_0 - t_1 = \int_0^{z_1}\frac{1}{H_0(1+z)\sqrt{\left[1+\Omega_{0,m}z\right](1+z)^2 - z\left[2+z\right]\Omega_\Lambda}}$$

Distance – In a different way

I have a standard source emitting a flash of light at the time t_{em}. If the photons emitted are N_{em} a telescope of Area A would receive, disregarding expansion and considering the physical distance is $a(t_{em})\, r$, $A / 4\pi (a(t_{em})\, r)^2$ photons.

However the Universe is expanding and at the time of detection the area of the sphere where the photons arrive as expanded to $4\pi (a(t_{obs})\, r)^2$.

That is $N_{em}/N_{detected} = A / (a(t_{obs})\, r)^2$

We have to take into account of two more factors:

1. Photons are redshifted so that their energy changes by a factor $1=z=a(t_{obs})/a(t_{em})$
2. The time between flashes δt at the source increases by a factor (1+z). This also corresponds to a loss of Energy per Unit Area of the telescope.

The Luminosity distance can be defined as:

$$Flux = \frac{L}{4\pi a^2(t_0)\, r^2 (1+z)^2} \equiv \frac{L}{4\pi d_L^2}$$

$$d_L = \sqrt{\frac{L}{4\pi Flux}} = a(t_0)\, r (1+z) \; for\ dust$$

$$a_0 r = \frac{2c}{H_0} \frac{\Omega_0 z + (\Omega_0 - 2)\left[-1 + (\Omega_0 z + 1)^{\frac{1}{2}}\right]}{\Omega_0^2 (1+z)}$$

$$z\ small \quad \sqrt{1+\Omega_0 z} = 1 + \frac{1}{2}\Omega_0 z$$

$$a_0 r = \frac{2c}{H_0} \frac{\Omega_0 z + \frac{1}{2}\Omega_0^2 z - \Omega_0 z}{\Omega_0^2 (1+z)} = \frac{c}{H_0} \frac{\Omega^2{}_0 z}{\Omega_0^2 (1+z)}$$

$$d_L = \frac{c}{H_0} z = \frac{v}{H_0}$$

Distance

$$d_L = a_0 r(1+z)$$

$$\frac{dr}{\sqrt{1-kr^2}} = \frac{cdt}{a(t)} = \frac{(1+z)cdt}{a_0} \text{ see slide look back time}$$

$$a_0 \int_0^r \frac{dr}{\sqrt{1-kr^2}} = (1+z)\, c\, dt = -\frac{c\, dz}{H_0 \sqrt{\left[\Omega_{0,R}(1+z)^4 + \Omega_{0,B}(1+z)^3 + \Omega_{0,DM}(1+z)^3 + \Omega_K (1+z)^2 + \Omega_\Lambda\right]}}$$

$$a_0 \int_0^r \frac{dr}{\sqrt{1-kr^2}} = \int_0^z \frac{c\, dz}{H_0 \sqrt{\left(\left[1+\Omega_{0,m}\right](1+z)^2 - z\left[2+z\right]\Omega_\Lambda\right)}}$$

$$\text{Mattig's formula } r = \frac{2c}{H_0 a_0} \frac{\Omega_0 z + (\Omega_0 - 2)\left[-1 + (\Omega_0 z + 1)^{\frac{1}{2}}\right]}{\Omega_0^{\,2}(1+z)}$$

$$\text{Extension to Matter + Radiation}$$

$$r = \frac{2c}{H_0 a_0} \frac{\Omega_m z + (\Omega_m + 2\Omega_R - 2)\left[-1 + \left(\Omega_m z + \Omega_R\left(z^2 + 2z\right)1\right)^{\frac{1}{2}}\right]}{\left[\Omega_m^{\,2} + 4\Omega_R(\Omega_R + \Omega_m - 1)\right](1+z)}$$

And with Λ

$$a_0 \int_0^r \frac{dr}{\sqrt{1-kr^2}} = \begin{cases} a_o \arcsin(r) & k=1 \\ a_0 r & k=0 \\ a_0 \arcsin h(r) & k=-1 \end{cases}$$

$$Example\ k=1\ and\ flat\ \Omega_{tot}=1$$

$$d_L = a_0 r (1+z) = \frac{c(1+z)}{H_0} \left\{ \int_0^z \frac{dz}{\sqrt{\left(\left[1+\Omega_{0,m}\, z\right](1+z)^2 - z\left[2+z\right]\Omega_\Lambda\right)}} \right\}$$

And for $\Omega_k < 0$

$$a_o \, arcsin(r) = \int_0^z \frac{c\,dz}{H_0\sqrt{\left(\left[1+\Omega_{0,m}z\right](1+z)^2 - z\left[2+z\right]\Omega_\Lambda\right)}} \quad or$$

$$r = sin\left\{\frac{c}{a_0 H_0}\int_0^z \frac{dz}{\sqrt{\left(\left[1+\Omega_{0,m}\,z\right](1+z)^2 - z\left[2+z\right]\Omega_\Lambda\right)}}\right\}$$

$$I\ use\ \Omega_k = -\frac{k\,c^2}{a_0{}^2 H_0^2} = 1 - \Omega_m - \Omega_\Lambda\ ;$$

$$d_L = a_0 r(1+z) = \frac{c\,(1+z)}{H_0\sqrt{|\Omega_k|}} sin\left\{\sqrt{|\Omega_k|}\int_0^z \frac{dz}{\sqrt{\left(\left[1+\Omega_{0,m}\,z\right](1+z)^2 - z\left[2+z\right]\Omega_\Lambda\right)}}\right\}$$

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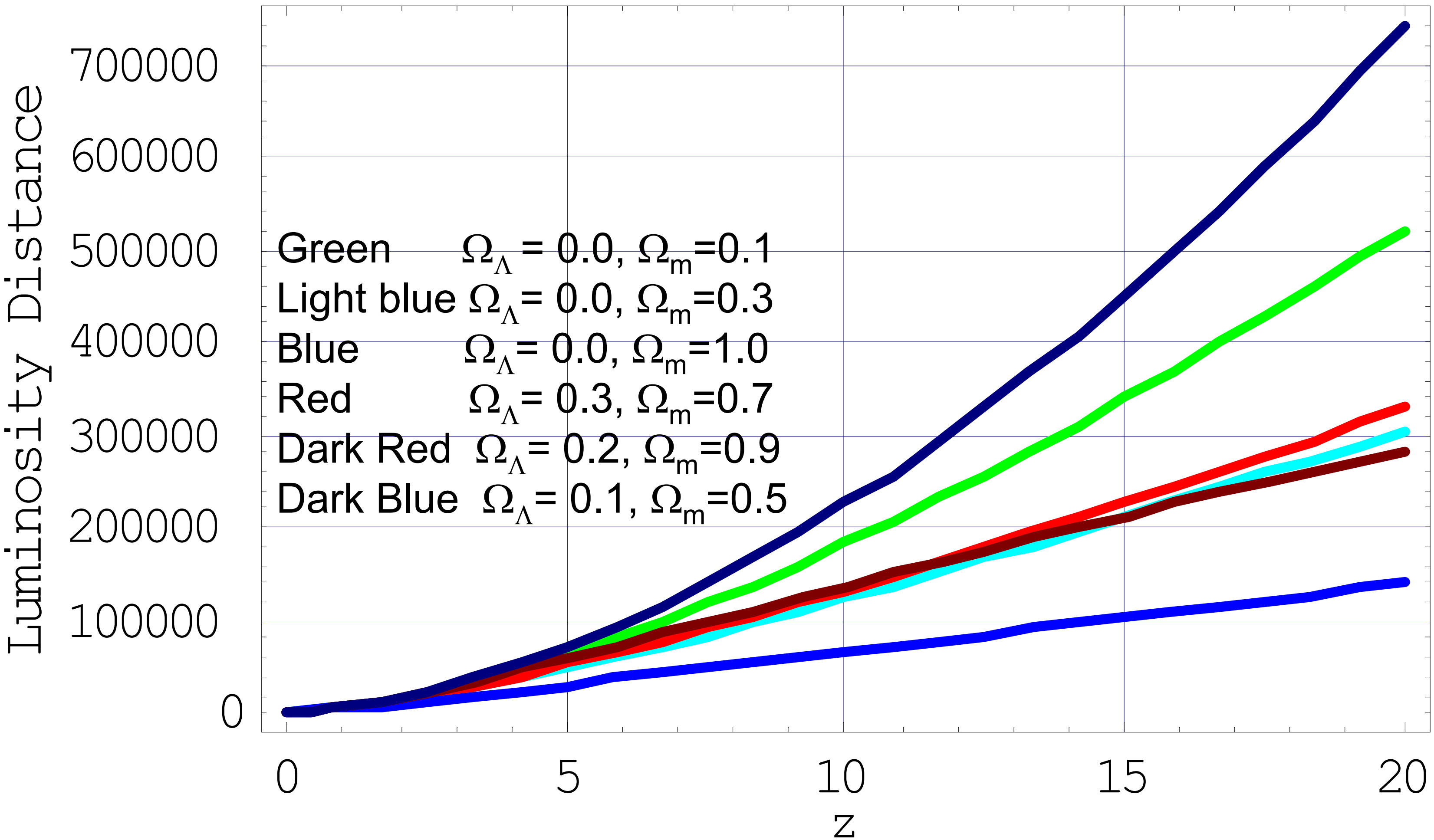
And for $\Omega_k > 0$ - Open

$$a_o \, arcsin\,h(r) = \int_0^z \frac{c\,dz}{H_0\sqrt{\left(\left[1+\Omega_{0,m}z\right](1+z)^2 - z\left[2+z\right]\Omega_\Lambda\right)}} \quad or$$

$$r = sinh\left\{\frac{c}{a_0 H_0}\int_0^z \frac{dz}{\sqrt{\left(\left[1+\Omega_{0,m}z\right](1+z)^2 - z\left[2+z\right]\Omega_\Lambda\right)}}\right\}$$

$$I\ use\ \Omega_k = -\frac{k\,c^2}{a_0{}^2 H_0^2} = 1 - \Omega_m - \Omega_\Lambda\ ;$$

$$d_L = a_0 r(1+z) = \frac{c\,(1+z)}{H_0\sqrt{|\Omega_k|}}\,sinh\left\{\sqrt{|\Omega_k|}\int_0^z \frac{dz}{\sqrt{\left(\left[1+\Omega_{0,m}z\right](1+z)^2 - z\left[2+z\right]\Omega_\Lambda\right)}}\right\}$$


Luminosity Distance
700000
600000
500000
400000
300000
200000
100000
0
0
5
10
15
20
z
Green $\Omega_\Lambda = 0.0$, $\Omega_m = 0.1$
Light blue $\Omega_\Lambda = 0.0$, $\Omega_m = 0.3$
Blue $\Omega_\Lambda = 0.0$, $\Omega_m = 1.0$
Red $\Omega_\Lambda = 0.3$, $\Omega_m = 0.7$
Dark Red $\Omega_\Lambda = 0.2$, $\Omega_m = 0.9$
Dark Blue $\Omega_\Lambda = 0.1$, $\Omega_m = 0.5$

Magnitudes

$$m_{Bol}(z) = M + 5\log d_L(z, H_0, \Omega_m, \Omega_\Lambda) + 25$$

Volume - Proper

$$\Delta^2\tau = c^2\Delta t^2 - a^2(t)\left\{\frac{\Delta^2 r}{\left(1-kr^2\right)} + r^2\left(\Delta\theta^2 + Sin^2\theta\,\Delta\phi^2\right)\right\}$$

$$Pr\,oper\;Volume :\Rightarrow d\Omega\left(or\;4\pi\right)\left(a_{em}r_{em}\right)^2\;dl\;;$$

$$dl \equiv line\;of\;\;sight\;dis\,tan\,ce\;travelled\;by\;\;photons\;in\;dz$$

$$dl = c\;dt = c\left(\frac{dt}{da}\right)\left(\frac{da}{dz}\right)\left(dz\right) = c\frac{1}{\dot a}\frac{a_0}{\left(1+z\right)^2}\frac{a}{a} = c\frac{a}{\dot a}\frac{a_0}{\left(1+z\right)^2}\frac{\left(1+z\right)}{a_0}$$

$$cdt = c\frac{1}{H}\frac{1}{\left(1+z\right)}dz$$

$$dV = d\Omega\frac{c\left(a_{em}r_{em}\right)^2}{H\left(1+z\right)}dz = d\Omega\frac{c\left(a_0\,r\right)^2}{H\left(1+z\right)^3}dz \Rightarrow dV = d\Omega\frac{c\,{d_L}^2}{H\left(1+z\right)^5}dz$$

Volume - Comoving

$$dV = d\Omega\,(or\ 4\pi)\left(a_0 r\right)^2 a_0 dr$$

$$a\,dr = c\,dt = \frac{c\,da}{\dot{a}} = \frac{c\,da}{a\,H}\ ;\ a_0 dr = \frac{a_0}{a}\,a\,dr = \left(1+z\right)\frac{c\,da}{a\,H} =$$

$$\frac{\left(1+z\right)c\,da}{H\,\frac{a_0}{\left(1+z\right)}} = \frac{c\,da}{H\,a_0}\left(1+z\right)^2 = \left\{since\ da = \frac{a_0\,dz}{\left(1+z\right)^2}\right\} = \frac{c}{H}dz$$

$$Assume\ matter + \Lambda\left(see\ earlier\ slide\right)$$

$$dV = d\Omega\,\frac{d_L^{\,2}}{\left(1+z\right)^2}\,\frac{c\,dz}{H_0\sqrt{\left[1+\Omega_{0,m}\,z\right]\left(1+z\right)^2 - z\left[2+z\right]\Omega_\Lambda}}$$

Comoving - Dust Universe

$$dV = d\Omega \frac{d_L^{\,2}}{(1+z)^2} \frac{c\,dz}{H_0 (1+z)\sqrt{\Omega_0\, z+1}} =$$

$$d_L = \frac{2c}{H_0} \frac{\Omega_0 z + (\Omega_0 - 2)\left[-1 + (\Omega_0 z + 1)^{\frac{1}{2}}\right]}{\Omega_0^{\,2}(1+z)}(1+z)$$

$$dV = d\Omega \frac{1}{(1+z)^2} \frac{c\,dz}{H_0 (1+z)\sqrt{\Omega_0\, z+1}} \left\{ \frac{2c}{H_0} \frac{\Omega_0 z + (\Omega_0 - 2)\left[-1 + (\Omega_0 z + 1)^{\frac{1}{2}}\right]}{\Omega_0^{\,2}(1+z)}(1+z) \right\}^2 =$$

$$dV = d\Omega \frac{c\,dz}{H_0^3 (1+z)^3} \frac{\left(\Omega_0 z + (\Omega_0 - 2)\left[-1 + (\Omega_0 z + 1)^{\frac{1}{2}}\right]\right)^2}{\Omega_0^{\,4}\sqrt{\Omega_0\, z+1}}$$

Ω_m Ω_Λ